approach with the standard approach. One difficulty which is present in this approach but not present in standard treatments arises in cases in which one is seeking a derivative free quadrature formula. The $m(n - 1)$ arbitrary parameters which occur in the choice of $\phi_j(x)$ have to be chosen to make the weights $A_{h,i} = O_j h \neq 0$. This introduces a linear constraint problem. Once this is solved or circumvented, the derivation of a quadrature formula follows the standard familiar pattern, with occasionally some minor variant.

The only 'new' formulae which I noticed consisted of a set due to Rebolia and Varna which were analogues of up to five-point Gaussian type formulae involving, besides function values, up to four derivatives at each point. Apart from this short collection, the authors apparently have not used their theory to provide any new quadrature formulae or to derive any hitherto undiscovered properties of known quadrature formulae.

To summarize, the part of this book which deals with the theory should be of interest to research workers in the field of numerical quadrature. The remainder of the book consists of a repository of special derivations of quadrature formulae using this new method, and may be of interest to research workers.

J. N. L.

29[2.20, 2.35, 3, 13.35].—E. POLAK, *Computational Methods in Optimization: A Unified Approach*, Academic Press, New York, 1971, xvii + 329 pp., 24 cm. Price $17.50.

The intent of this detailed book is to present in a unified manner almost all of the important algorithms invented to date for solving nonlinear programming, optimal control, root finding, and boundary value problems. The first chapter contains the statements of the problems to be solved, the John, Kuhn-Tucker, and Pontryagin conditions which characterize solutions to these problems, and an exposition of simple prototype models of algorithms for solving these problems. The second chapter deals with methods for finding points in R^n which minimize continuously differentiable functions and then applies these to the problem of finding solutions to unconstrained discrete optimal control and unconstrained continuous optimal control problems. Included are the methods of steepest descent, golden section search (for unconstrained problems in one variable), Newton-Raphson, local variations, conjugate gradients, variable metric (Davidon-Fletcher-Powell), and a modified quasi-Newton method of the author's. Several modifications of the above methods are described. In Chapter 3, the Newton-Raphson method for solving nonlinear equations is used as a basis for algorithms to solve equality constrained optimization problems in R^n, boundary value and discrete optimal control problems, and boundary value and continuous optimal control problems.

Algorithms for solving the general nonlinear programming problem with inequality and equality constraints are described in Chapter 4. Interior and exterior penalty function methods, methods of centers, methods of feasible directions, and gradient projection methods are covered. The use of some of these methods for solving optimal control problems is explained. Chapter 5 deals with discrete optimal control problems which can be cast as convex nonlinear programming problems. Some of these are shown to be solvable by standard linear and quadratic programming techniques. A dual decomposition algorithm is described for solving a restricted class of convex optimal control problems which can be transcribed into a 'primal problem.' A more general class of problems, 'geometric problems,' is defined, and a primal decomposition algorithm for solving these is explained. The final chapter gives theorems on the rates of convergence of the steepest descent, Newton-Raphson, conjugate gradient, and variable metric methods for minimizing an unconstrained function. The rates of convergence of the conjugate gradient and variable metric methods have been only recently proved (by A. Cohen and M. J. D. Powell, respectively) and their inclusion makes this chapter very current.

Three concluding appendices contain generalizations of the author's ideas on the implementation of conceptual algorithms, background material on properties of continuous and convex functions, and a summary of the author's preferences for implementable versions of many of the algorithms presented in the book.

Most students using this book as a text would face several difficulties. The writing is generally good, and some explanations (such as that on the reason why the method of steepest descent works poorly) are illuminating, but the author uses hardly any examples (there are no small illustrative examples with numbers in the book). The exercises, some of which are intriguing for advanced researchers who already know the material, are no aid to learning since they consist almost entirely of requests to complete uncompleted proofs or generalize already extended concepts or algorithms. The exposition contains much material which is not explained but could easily have been explained in Appendix B. The appendix covers convex functions but makes no mention of convex sets. The text contains such terms as 'strictly convex set' and uses the fundamental separation theorem for convex sets. Topological concepts such as Banach spaces and Hilbert spaces appear without definition, usually when such generality is unwarranted.

The exposition is uneven and often emphasizes and complicates small points while ignoring or failing to make clear significant ones. For example, in Section 1.2, he discusses optimality conditions. Because the concept of a local minimizer is not brought up, the reader is not informed that the John conditions apply to other than global minimizers. He omits the second order necessary and second order sufficiency conditions, thereby failing to inform the reader of an important subject matter of optimization theory (and practice). In the statement of the John theorem (without proof), the feasibility requirement on the optimal point is omitted, the notation does not show the association of the multipliers with the particular optimal point, and it is not at all clear what the phrase "not all zero" modifies. Then, an interesting but minor proposition is proved using terms 'ray,' 'cone,' and 'separated' when it could have been proved directly from the John theorem.

There is a serious question whether the author's material on 'simple' prototype models makes it easier for the student to learn the variety of algorithms presented later.

The instructor, in addition to explaining the material in the book to the student, supplying examples and exercises, must also supply additional material to ensure full coverage of the growing field of optimization theory. Two key omissions are the generalized reduced gradient method and the combined gradient projection-variable metric method. These are two of the most successful algorithms implemented to date for solving nonlinear programming problems. Methods of feasible directions, on which the author spends much time, have been quite unsuccessful for nonlinear problems.

The instructor will also have to unteach much of the vocabulary learned from the book. The author uses 'quasi-Newton' to mean Newton-Raphson type algorithms. The term 'quasi-Newton' has come to apply to first order iteration schemes for approximation of the inverse Hessian matrix. Most people use 'methods of conjugate directions' for the class of algorithms the author calls 'conjugate gradient methods.' The conjugate gradient method is a particular member of this class. The author also includes, without discussion, the variable metric method as a member of this class although it should be obvious from the proof by Powell that it owes much of the superlinear rate of convergence characteristics to its 'quasi-Newton' character. The author never explains the principles behind the methods of conjugate directions, using as a prototype algorithm a useless general statement.

Finally, the instructor will have to disabuse the student of almost all the opinions about what constitutes a 'good' 'implementable' algorithm. Suggestions abound for parameter selection. These selections seem to be drawn from extensive computer efforts on problems with two variables. Each implementable algorithm has at least two parameters so one can probably obtain very good performance indeed on such problems. In statements like "Generally, exterior penalty function methods are considered to perform better than interior penalty function methods . . ." the author strays from his personal opinions and implies some concensus on this point. The general concensus is just the opposite. In fact, the author's earlier reasoned discussion tends to contradict this.

Another way of looking at this book is for its value as a reference or research monograph. Generally speaking, the author has been meticulous in his proofs of theorems and the book is almost error free. Where the author falls down is in the theorem statements, the hypotheses and conclusions. For one example, consider the convergence proofs for exterior penalty function methods. For the problem: minimize $f^0(z)$ subject to $z \in C \subset R^n$, he assumes that the set $\{z | f^0(z) \leqq f^0(z^1)\}$ is compact for some $z^1 \in C$. This makes the proving of convergence very easy but avoids all the hard questions such as what happens when the exterior penalty function method is applied to a problem when this assumption does not hold (and it usually does not). There are interesting answers to this question involving convergence to compact sets of local minimizers and, for the convex case, proofs of the boundedness of penalty function contours deriving from compact set solutions of the original problem. In short, the author leaves out much interesting fundamental material.

For another example, in the proof of the convergence of the variable metric method and its rate of convergence by Powell, the author has confused to a high degree the hypotheses of both portions of the paper. On p. 46, he presents a confused statement about the uniform bounds required on the eigenvalues of the Hessian matrix which is required for Powell's convergence proof. On p. 58, he states a Lipschitz condition which should be required for the rate of convergence proof and later on p. 269 he changes that to the assumption that $f^0 \in C^3$. Since one of the fascinating aspects of Powell's paper was the level of assumptions needed, it is important to reproduce these correctly.

The author makes much of his use of algorithm models as an aid in unifying the presentation of algorithms and their proofs. It would have been most interesting to see if his approach is more general than that of Zangwill who did the first extensive work in this area. Instead, we are given the sentence, "The following set of assumptions are due to Zangwill [Z1] and can be shown, though not very easily, to be stronger than [my assumptions] . . .". Such distinctions are the meat of research and it is very important not to omit proofs of such statements.

Finally, it is very amusing after the author has written so much about the importance of proposing 'implementable' as opposed to 'conceptual' algorithms to read the following step in at least seven of his 'implementable' algorithms for minimizing an unconstrained function $f^0(z)$. "Step 0. Select a $z_0 \in R^n$ such that the set $C(z_0) = \{z | f^0(z) \leqq f^0(z_0)\}$ is bounded."

This book is important because of the breadth of material it contains. The chapter on the rate of convergence of unconstrained minimization techniques is very up-to-date. For these reasons, it is a useful addition to anyone's library.

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30[2.20, 2.40].—CHIH-BING LING & JUNG LIN, *Values of Coefficients in Problems of Rotational Symmetry*, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, and Tennessee Technological University, Cookeville, Tennessee, February 1972, ms. of 23 typewritten pages deposited in the UMT file.

The finite difference $\Delta^s \sigma_n$ arises frequently in problems of rotational symmetry, where σ_n is the sum of the nth powers of the roots of the equation $u^k - (u - 1)^k = 0$, $k \geqq 2$. In general, σ_n is real.

The authors tabulate $\Delta^s \sigma_n$ to 11S for $k = 3(1)8$, with $n = -4(1)65$ and $s = 0(1)k - 1$. For $s \geqq k$, values of the differences can be found from the tabulated values by the relation $\Delta^s \sigma_n = \Delta^{s-mk} \sigma_{n+mk}$, where m is a positive integer such that $0 \leqq s - mk \leqq k - 1$. In particular, for $k = 2$ we have $\Delta^s \sigma_n = (-1)^s / 2^{n+s}$.

AUTHORS' SUMMARY

31[3].—STEFAN FENYŐ, *Moderne Mathematische Methoden in der Technik*, Vol. II, Birkhäuser Verlag, Basel, 1971, 336 pp., 25 cm. Price 62—Fr.

This second volume, in contrast to the first, may be described as dealing with finite methods in applied mathematics. In three chapters, it covers linear algebra, linear and convex programming, and graph theory. While the first two chapters would offer ample opportunity for including computational considerations, the author deliberately omits such topics. He feels that their inclusion would lead beyond