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Remark on a Paper by Huddleston

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Abstract. Using a function-theoretic approach, we obtain, in a quite simple way, linear relations between the values of a function and its first derivatives at n abscissa points $x_1, \cdots, x_n$. The derivation of these formulae in a recent paper by Huddleston was rather cumbersome. Possible generalizations are indicated.

1. In a recent paper, Huddleston [1] gave some relations between the values of a function and its first derivatives at n abscissa points. Huddleston's derivation is, to speak with his own words, "an exercise in drudgery". Using a function theoretic approach, we give a new proof of the results in [1] which is both simple and lucid and, in addition, indicates how one may obtain more general relations by the same method.

2. Let C be a simple closed rectifiable positively oriented curve in the complex plane. Let $x_1 < x_2 < \cdots < x_n$ be n points on the real axis which lie in the interior of C, and let

$$w(z) = (z - x_1)(z - x_2) \cdots (z - x_n).$$

For functions $f(z)$, holomorphic in a domain G which contains C, consider the linear functional

$$L(f) = \frac{1}{2\pi i} \int_C \frac{f(z)}{w^2(z)}\, dz. \tag{1}$$

Clearly, $L(f)$ vanishes if $f(z)$ is a polynomial $P_{2n-2}(z)$ of degree less than or equal to $2n - 2$. From the Taylor series

$$f(z) = f(x_\nu) + f'(x_\nu)(z - x_\nu) + \cdots$$

and

$$w^2(z) = w'^2(x_\nu)(z - x_\nu)^2 + w'(x_\nu)w''(x_\nu)(z - x_\nu)^3 + \cdots,$$

we get

$$\operatorname*{res}_{z=x_\nu} \frac{f(z)}{w^2(z)} = \frac{1}{w'^2(x_\nu)} f'(x_\nu) - \frac{w''(x_\nu)}{w'^3(x_\nu)} f(x_\nu), \tag{2}$$

and the residue theorem gives

$$L(f) = \sum_{\nu=1}^{n} \frac{1}{w'^2(x_\nu)} f'(x_\nu) - \frac{w''(x_\nu)}{w'^3(x_\nu)} f(x_\nu).$$

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For $f(z) = P_{2n-2}(z)$, we obtain Huddleston's formula

$$\sum_{\nu=1}^{n} \frac{1}{w'^2(x_\nu)} P'_{2n-2}(x_\nu) - \frac{w''(x_\nu)}{w'^3(x_\nu)} P_{2n-2}(x_\nu) = 0.$$

3. Using the fact that $L(f)$ is equal to the divided difference with coalescent knots $[x_1x_1x_2x_2 \cdots x_nx_n]$ (see [2, p. 199]), we get in the case that $f(z)$ is real for real z:

$$L(f) = \frac{f^{(2n-1)}(\xi)}{(2n-1)!}, \qquad \xi \in (x_1, x_n) \tag{3}$$

(see [2, p. 13]). Huddleston's formula (5.1), (5.2) is a consequence of (1), (2) and (3).

4. In the case of equidistant knots, e.g. $x_\nu = \nu$, $\nu = 1(1)n$, we arrive at

$$\sum_{\nu=1}^{n} \binom{n-1}{\nu-1}^2 \left[f'(\nu) - \sum_{\mu=1;\mu\neq\nu}^{n} \frac{2}{\nu-\mu} f(\nu) \right] = \frac{[(n-1)!]^2}{(2n-1)!} f^{(2n-1)}(\xi).$$

5. Obviously, our method may be generalized to obtain similar relations for other Hermite data.

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1. R. E. Huddleston, "Some relations between the values of a function and its first derivative at n abscissa points," *Math. Comp.*, v. 25, 1971, pp. 553–558.
2. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.