MATHEMATICS OF COMPUTATION, VOLUME 26, NUMBER 120, OCTOBER 1972

The Condition of Orthogonal Polynomials

By Walter Gautschi

Abstract. An estimate is given for the condition number of the coordinate map associating to each polynomial its coefficients with respect to a system of orthogonal polynomials.

Let $w(x) \geqq 0$ be a weight function on the finite interval $[a, b]$, and $\{p_k(x)\}_{k=0}^{\infty}$ the associated orthogonal polynomials. We consider the linear parametrization map M_n: $\mathbf{R}^n \to \mathbf{P}_{n-1}$ which associates to each (real) vector $u^T = [u_0, u_1, \cdots, u_{n-1}] \in \mathbf{R}^n$ the (real) polynomial $p(x) = \sum_{k=0}^{n-1} u_k p_k(x) \in \mathbf{P}_{n-1}$. The object of this note is to estimate the condition

$$\operatorname{cond}_\infty M_n = ||M_n||_\infty \, ||M_n^{-1}||_\infty$$

of the map M_n, the infinity norms in $\mathbf{R}^n$ being defined by $||u||_\infty = \max_{0 \leqq k \leqq n-1} |u_k|$, and in $\mathbf{P}_{n-1}$ by $||p||_\infty = \max_{a \leqq x \leqq b} |p(x)|$. Letting

$$\mu_0 = \int_a^b w(x)\,dx, \qquad h_k = \int_a^b p_k^2(x) w(x)\,dx, \qquad k = 0, 1, 2, \cdots,$$

we show in fact that

$$\operatorname{cond}_\infty M_n \leqq \max_{0 \leqq k \leqq n-1} \left(\frac{\mu_0}{h_k}\right)^{1/2} \max_{a \leqq x \leqq b} \sum_{k=0}^{n-1} |p_k(x)|. \tag{1}$$

For Chebyshev polynomials $p_k(x) = T_k(x)$ on $[-1, 1]$, e.g., this gives

$$\operatorname{cond}_\infty M_n \leqq 2^{1/2} n \qquad (p_k = T_k),$$

while for Legendre polynomials $p_k(x) = P_k(x)$ on $[-1, 1]$ one gets

$$\operatorname{cond}_\infty M_n \leqq n(2n-1)^{1/2} \qquad (p_k = P_k).$$

In order to prove (1), we first observe that, for any $u \in \mathbf{R}^n$,

$$||M_n u||_\infty = \left\| \sum_{k=0}^{n-1} u_k p_k(x) \right\|_\infty \leqq ||u||_\infty \max_{a \leqq x \leqq b} \sum_{k=0}^{n-1} |p_k(x)|,$$

so that

$$||M_n||_\infty \leqq \max_{a \leqq x \leqq b} \sum_{k=0}^{n-1} |p_k(x)|. \tag{2}$$

On the other hand, if $M_n^{-1} p = u$, then, by orthogonality,

Received December 27, 1971.

AMS 1970 *subject classifications.* Primary 33A65; Secondary 65G05.

Key words and phrases. Orthogonal polynomials, parametrization, conditioning, Chebyshev polynomials, Legendre polynomials.

$$u_k = \frac{1}{h_k}\int_a^b p(x)p_k(x)w(x)\,dx, \qquad k = 0, 1, \cdots, n-1.$$

Therefore, using the Schwarz inequality,

$$\begin{aligned} |u_k| &\leqq \frac{1}{h_k}\int_a^b |p(x)|\,(w(x))^{1/2}\cdot|p_k(x)|\,(w(x))^{1/2}\,dx \\ &\leqq \frac{1}{h_k}\left(\int_a^b p^2(x)w(x)\,dx \int_a^b p_k^2(x)w(x)\,dx\right)^{1/2} \\ &\leqq \frac{1}{h_k}\left(\|p\|_\infty^2 \int_a^b w(x)\,dx\cdot h_k\right)^{1/2} = \|p\|_\infty(\mu_0/h_k)^{1/2}. \end{aligned}$$

It follows that, for all $p \in \mathbf{P}_{n-1}$,

$$\|M_n^{-1}p\|_\infty \leqq \|p\|_\infty \max_{0\leqq k\leqq n-1} (\mu_0/h_k)^{1/2},$$

so that

$$\|M_n^{-1}\|_\infty \leqq \max_{0\leqq k\leqq n-1} (\mu_0/h_k)^{1/2}. \tag{3}$$

Combining (2) and (3) gives the desired result (1).

In terms of the orthonormal polynomials $\pi_k(x) = h_k^{-1/2}p_k(x)$, we may write (1) in the form

$$\operatorname{cond}_\infty M_n \leqq \max_{0\leqq k\leqq n-1} (\mu_0/h_k)^{1/2} \max_{a\leqq x\leqq b} \sum_{k=0}^{n-1} h_k^{1/2}\,|\pi_k(x)|. \tag{1'}$$

If we let $h = \min_{0\leqq k\leqq n-1} h_k$, we see that the bound in (1′) is larger than or equal to

$$(\mu_0/h)^{1/2} \max_{a\leqq x\leqq b} \sum_{k=0}^{n-1} h^{1/2}\,|\pi_k(x)| = \mu_0^{1/2} \max_{a\leqq x\leqq b} \sum_{k=0}^{n-1} |\pi_k(x)|,$$

so that, among all possible normalizations, the one with $h_0 = h_1 = \cdots = h_{n-1}$ gives the best bound in (1).

Acknowledgment. The author is indebted to the referee for the observation made in the last sentence of this note.

Department of Computer Sciences
Purdue University
Lafayette, Indiana 47907