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On the General Hermite Cardinal Interpolation

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Abstract. A sequence of interpolation series is given which generalizes Whittaker's cardinal function to the case of Hermite interpolation. By integrating the interpolation series, a sequence of new quadrature formulae for $\int_{-\infty}^{\infty} f(x)\,dx$ is obtained. Derivative-free remainders are stated for these interpolation and quadrature formulae.

Given a function $f : \mathbf{R} \to \mathbf{C}$ and a real number $h > 0$, the series

$$T_h(f)(z) := \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(mh)}{z - mh} \sin \frac{\pi}{h} z$$

$$= \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{f(mh)}{z - mh} \sin \frac{\pi}{h} (z - mh), \qquad z \in \mathbf{C},$$

is called the *cardinal series* of the function f with respect to the interval h. If the series converges, its sum $T_h(f)$ is called the *cardinal function* or *cardinal interpolation* of the function f. Obviously,

$$T_h(f)(mh) = f(mh), \qquad m = 0, \pm 1, \pm 2, \cdots,$$

holds. In the case when $f : B \to \mathbf{C}$ is analytic in a strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$, and satisfies certain conditions at infinity, a derivative-free remainder for this cardinal interpolation was independently found by Kress [2] and McNamee, Stenger and Whitney [6].

In the present paper, we generalize the cardinal interpolation and give a sequence of *Hermite cardinal interpolations* $T_{p,h}(f)$, $p = 0, 1, 2, \cdots$, with

$$T_{p,h}^{(q)}(f)(mh) = f^{(q)}(mh), \qquad q = 0, 1, \cdots, p, \quad m = 0, \pm 1, \pm 2, \cdots.$$

The usual cardinal interpolation is included as the particular case $p = 0$.

In Section 1, we give the explicit form of $T_{p,h}(f)$ and state a derivative-free remainder. Making use of this remainder, we describe a class of functions for which $T_{p,h}(f) = f$.

In Section 2, we apply the general cardinal functions to derive a sequence $I_{p,h}(f)$, $p = 0, 2, 4, \cdots$, of integration formulae for infinite integrals involving the derivatives $f^{(q)}(mh)$, $q = 0, 2, \cdots, p$, $m = 0, \pm 1, \pm 2, \cdots$, which may be regarded as generalizations of the trapezoidal rule. The remainder, given by Goodwin [1], Martensen [4] and McNamee [5] for the trapezoidal rule, is extended to the quadrature formulae $I_{p,h}(f)$.

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The general cardinal interpolation developed in this paper is closely related to the general Hermite trigonometric interpolation of periodic functions [3].

1. Interpolation. Let $p \geqq 0$ be an integer and let $h > 0$ be real. Define $p + 1$ entire functions t_q, $q = 0, 1, \cdots, p$, by

$$t_q(z) := \frac{z^q}{q!}\left(\frac{\sin(\pi/h)z}{(\pi/h)z}\right)^{p+1} \sum_{r=0;\, r \text{ even}}^{2[(p-q)/2]} a_r(p)\left(\frac{\pi}{h} z\right)^r, \qquad z \in \mathbf{C}, \tag{1.1}$$

where the $a_r(p)$ are the coefficients of the Laurent expansion

$$\frac{1}{\sin^{p+1} z} = \sum_{r=0;\, r \text{ even}}^{\infty} \frac{a_r(p)}{z^{1+p-r}}, \qquad 0 < |z| < \pi. \tag{1.2}$$

To avoid indexing difficulties, we do not indicate the dependence of the t_q on p and h.

LEMMA 1.1. *For every $r = 0, 1, \cdots, p$, the functions t_q, $q = 0, 1, \cdots, p$, satisfy*

$$t_q^{(r)}(0) = \delta_q^r, \tag{1.3}$$

$$t_q^{(r)}(mh) = 0, \qquad m = \pm 1, \pm 2, \cdots. \tag{1.4}$$

Proof. From (1.1) and (1.2), we obtain

$$t_q(z) = z^q/q! + z^{p+1}u_q(z), \qquad q = 0, 1, \cdots, p,$$

with certain entire functions u_q, and (1.3) immediately follows. The relation (1.4) trivially holds.

Definition 1.1. *Let $p \geqq 0$ be an integer and let $h > 0$ be real. Given a function $f: \mathbf{R} \to \mathbf{C}$, $f \in C^p(\mathbf{R})$, the pth cardinal series of f with respect to the interval h is defined by*

$$T_{p,h}(f)(z) := \sum_{m=-\infty}^{\infty} \sum_{q=0}^{p} f^{(q)}(mh)t_q(z - mh), \qquad z \in \mathbf{C}. \tag{1.5}$$

If the series converges, its sum $T_{p,h}(f)$ is called the pth cardinal function of f.

Lemma 1.1 implies

THEOREM 1.1. *The pth cardinal function $T_{p,h}(f)$ is a Hermite interpolation of the function f with equidistant interpolation points*

$$T_{p,h}^{(q)}(f)(mh) = f^{(q)}(mh), \qquad q = 0, 1, \cdots, p, \quad m = 0, \pm 1, \cdots. \tag{1.6}$$

The first cardinal series is listed below.

$$T_{0,h}(f)(z) = \frac{h}{\pi} \sum_{m=-\infty}^{\infty} (-1)^m \frac{f(mh)}{z - mh} \sin\frac{\pi}{h} z,$$

$$T_{1,h}(f)(z) = \left(\frac{h}{\pi}\right)^2 \sum_{m=-\infty}^{\infty} \left\{\frac{f(mh)}{(z - mh)^2} + \frac{f'(mh)}{z - mh}\right\} \sin^2\frac{\pi}{h} z,$$

$$T_{2,h}(f)(z) = \left(\frac{h}{\pi}\right)^3 \sum_{m=-\infty}^{\infty} (-1)^m \left\{\frac{f(mh)}{(z - mh)^3} - \frac{1}{2}\left(\frac{\pi}{h}\right)^2 \frac{f(mh)}{z - mh} + \frac{f'(mh)}{(z - mh)^2} + \frac{f''(mh)}{z - mh}\right\} \sin^3\frac{\pi}{h} z.$$

In the case when the function f is analytic in a strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$, we shall give a sufficient condition on the convergence of the pth cardinal series of f and shall obtain a representation of the remainder

$$R_{p,h}(f) := f - T_{p,h}(f). \tag{1.7}$$

LEMMA 1.2. *Let the function f be analytic in the strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$. Then*

$$\begin{aligned} \chi_n(z) f(z) - \sum_{m=-n}^{n} \sum_{q=0}^{p} f^{(q)}(mh) t_q(z - mh) \\ = \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} z \int_{C_n} \frac{f(\zeta)\, d\zeta}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta}, \quad z \notin C_n, \end{aligned} \tag{1.8}$$

where C_n denotes the boundary of a rectangle $B_n := [-(n + \frac{1}{2})h, (n + \frac{1}{2})h] \times [-a, a] \subset B$ and where χ_n denotes the characteristic function of B_n with $\chi_n(z) = 1$, $z \in B_n$ and $\chi_n(z) = 0$, $z \notin B_n$.

Proof. The function $F: B_n \to \mathbf{C}$, defined by

$$F(z) := \frac{1}{\sin^{p+1}(\pi/h)z} \left(f(z) - \sum_{m=-n}^{n} \sum_{q=0}^{p} f^{(q)}(mh) t_q(z - mh) \right), \quad z \in B_n, \tag{1.9}$$

is analytic. Hence, by Cauchy's theorem,

$$\chi_n(z) F(z) = \frac{1}{2\pi i} \int_{C_n} \frac{F(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin C_n. \tag{1.10}$$

Using the identities

$$\int_{C_n} \frac{d\zeta}{(\zeta - z)(\zeta - mh)^{q+1}} = \frac{-2\pi i}{(z - mh)^{q+1}} (1 - \chi_n(z)), \quad z \notin C_n,$$

$q = 0, 1, \cdots, p$, $m = 0, \pm 1, \cdots, \pm n$, we substitute (1.9) into (1.10) and obtain (1.8).

THEOREM 1.2. *Let the function f be analytic and bounded in the strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$, and let*

$$\int_{-\infty - ia}^{\infty - ia} |f(z)|^2\, ds < \infty, \qquad \int_{-\infty + ia}^{\infty + ia} |f(z)|^2\, ds < \infty. \tag{1.11}$$

Then, for arbitrary $p \geqq 0$ and $h > 0$, the pth cardinal series of f with respect to the interval h is locally uniformly convergent for all $x \in \mathbf{R}$ and the remainder (1.7) is given by

$$\begin{aligned} R_{p,h}(f)(x) = \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} x \left\{ \int_{-\infty - ia}^{\infty - ia} \frac{f(\zeta)\, d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right. \\ \left. - \int_{-\infty + ia}^{\infty + ia} \frac{f(\zeta)\, d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right\}, \quad x \in \mathbf{R}, \end{aligned} \tag{1.12}$$

and bounded by

$$\begin{aligned} |R_{p,h}(f)(x)| \leqq \frac{1}{2(\pi a)^{1/2}} \left(\frac{\sin(\pi/h)x}{\sinh(\pi/h)a} \right)^{p+1} \left\{ \left(\int_{-\infty - ia}^{\infty - ia} |f(\zeta)|^2\, ds \right)^{1/2} \right. \\ \left. + \left(\int_{-\infty + ia}^{\infty + ia} |f(\zeta)|^2\, ds \right)^{1/2} \right\}, \quad x \in \mathbf{R}. \end{aligned} \tag{1.13}$$

Proof. Let f be bounded by M. Then we estimate the integrals

$$\left|\int_{\pm(n+1/2)h-ia}^{\pm(n+1/2)h+ia} \frac{f(\zeta)\,d\zeta}{(\zeta - x)\sin^{p+1}(\pi/h)\zeta}\right| \leqq \frac{2Ma}{(n+\frac{1}{2})h - |x|}.$$

Thus, by Lemma 1.2,

$$f(x) = \lim_{n\to\infty}\left[\sum_{m=-n}^{n} f^{(q)}(mh)t_q(x - mh)\right.$$

$$(1.14) \qquad + \frac{1}{2\pi i}\sin^{p+1}\frac{\pi}{h}x\left\{\int_{-(n+1/2)h-ia}^{+(n+1/2)h-ia} \frac{f(\zeta)\,d\zeta}{(\zeta - x)\sin^{p+1}(\pi/h)\zeta}\right.$$

$$\left.\left. - \int_{-(n+1/2)h+ia}^{+(n+1/2)h+ia} \frac{f(\zeta)\,d\zeta}{(\zeta - x)\sin^{p+1}(\pi/h)\zeta}\right\}\right], \qquad x \in \mathbf{R},$$

where convergence is locally uniform for all $x \in \mathbf{R}$. Upon noting that

$$\left|\sin\frac{\pi}{h}\zeta\right| \geqq \sinh\frac{\pi}{h}a, \qquad \zeta = \xi \pm ia,$$

and

$$\int_{-\infty\pm ia}^{\infty\pm ia} ds/|\zeta - x|^2 = \pi/a,$$

Schwarz's inequality yields

$$\left|\int_{-\infty\pm ia}^{\infty\pm ia} \frac{f(\zeta)\,d\zeta}{(\zeta - x)\sin^{p+1}(\pi/h)\zeta}\right| \leqq \frac{\sqrt{\pi}}{\sqrt{a}\sinh^{p+1}(\pi/h)a}\left(\int_{-\infty\pm ia}^{\infty\pm ia} |f(\zeta)|^2\,ds\right)^{1/2}, \qquad x \in \mathbf{R}.$$

Hence, letting $n \to \infty$ in (1.14) completes the proof.

Remark. From the bound (1.13), *we easily see that*

$$\lim_{h\to 0} T_{p,h}(f)(x) = f(x), \qquad p = 0, 1, \cdots,$$

and

$$\lim_{p\to\infty} T_{p,h}(f)(x) = f(x), \qquad h > 0, \quad \sinh\frac{\pi}{h}a > 1,$$

where convergence is uniform for all $x \in \mathbf{R}$. In both cases $h \to 0$, p fixed and $p \to \infty$, h fixed, the convergence is exponential.

The following theorem describes a class of functions for which $T_{p,h}(f) = f$ is true.

THEOREM 1.3. *Let f be an entire function, such that*

$$(1.15) \qquad |f(z)| \leqq ce^{\rho|y|}, \qquad z = x + iy \in \mathbf{C},$$

with real numbers $c \geqq 0$ and $0 \leqq \rho < (p+1)\pi/h$. Then the pth cardinal series for f with respect to h is locally uniformly convergent for all $z \in \mathbf{C}$, and the identity

$$(1.16) \qquad T_{p,h}(f)(z) = f(z), \qquad z \in \mathbf{C},$$

holds.

Proof. By (1.15) we have

$$\left|\frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta}\right| \leqq \frac{ce^{\rho|\eta|}}{\sinh^{p+1}(\pi/h)\,|\eta|}$$

$$= O\left(\exp\left[-\left[(p+1)\frac{\pi}{h}-\rho\right]|\eta|\right]\right), \qquad \zeta = \xi + i\eta \in \mathbf{C},$$

as $|\eta| \to \infty$, and therefore

$$\lim_{a\to\infty} \int_{-(n+1/2)h\pm ia}^{(n+1/2)h\pm ia} \frac{f(\zeta)\,d\zeta}{(\zeta - x)\sin^{p+1}(\pi/h)\zeta} = 0,$$

that is, by Lemma 1.2,

$$f(z) = \sum_{m=-n}^{n} \sum_{q=0}^{p} f^{(q)}(mh)t_q(z - mh)$$

$$(1.17) \qquad + \frac{1}{2\pi i}\sin^{p+1}\frac{\pi}{h}z\left\{\int_{(n+1/2)h-i\infty}^{(n+1/2)h+i\infty} \frac{f(\zeta)}{(\zeta - z)\sin^{p+1}(\pi/h)\zeta}\,d\zeta - \int_{-(n+1/2)h-i\infty}^{-(n+1/2)h+i\infty} \frac{f(\zeta)}{(\zeta - z)\sin^{p+1}(\pi/h)\zeta}\,d\zeta\right\},$$

$$z = x + iy \in \mathbf{C}$$

for all n with $(n + \frac{1}{2})h > |x|$. Making use of

$$\left|\sin\frac{\pi}{h}\zeta\right| = \cosh\frac{\pi}{h}\eta \geqq \frac{1}{2}\exp\left[\frac{\pi}{h}|\eta|\right], \qquad \zeta = \pm(n + \tfrac{1}{2})h + i\eta,$$

we conclude that

$$\left|\int_{\pm(n+1/2)h-i\infty}^{\pm(n+1/2)h+i\infty} \frac{f(\zeta)\,d\zeta}{(\zeta - z)\sin^{p+1}(\pi/h)\zeta}\right| \leqq \frac{2^{p+2}c}{(p+1)(\pi/h) - \rho}\,\frac{1}{(n+\frac{1}{2})h - |x|}.$$

Letting $n \to \infty$ in (1.17), the assertion of the theorem follows.

Example. If we choose $f(z) := e^{i\rho z}$, $z \in \mathbf{C}$, $0 \leqq \rho < (p + 1)\pi/h$, we obtain the local uniform convergent expansion

$$(1.18) \qquad e^{i\rho z} = \sum_{m=-\infty}^{\infty} \sum_{q=0}^{p} (i\rho)^q e^{i\rho mh} t_q(z - mh), \qquad z \in \mathbf{C}.$$

Setting $\rho := r\pi/h$, $r = 0, 1, \cdots, p$, we derive

$$(1.19) \qquad \exp\left[ir\frac{\pi}{h}z\right] = \sum_{m=-\infty}^{\infty} \sum_{q=0}^{p} \left(ir\frac{\pi}{h}\right)^q (-1)^{rm} t_q(z - mh), \qquad z \in \mathbf{C}.$$

2. Numerical Integration. We integrate the pth cardinal series of f termwise and obtain the series

$$(2.1) \qquad I_{p,h}(f) := h \sum_{m=-\infty}^{\infty} \sum_{q=0;\,q\text{ even}}^{p} \left(\frac{h}{2\pi}\right)^q a_{q,p} f^{(q)}(mh)$$

with the weights

$$(2.2) \qquad a_{q,p} := \frac{1}{2\pi}\left(\frac{2\pi}{h}\right)^{q+1} \int_{-\infty}^{\infty} t_q(x)\,dx, \qquad q = 0, 2, \cdots, p.$$

If q is odd, the integral (2.2) vanishes, since in this case the function t_q is odd. The series (2.1) may be regarded as generalizations of the trapezoidal rule approximation for the integral $\int_{-\infty}^{\infty} f(x)\,dx$.

In order to derive simple recurrence formulae for the weights $a_{q,p}$, we state

THEOREM 2.1. *Let p be even. Then the weights $a_{q,p}$ are uniquely determined by the identity*

$$\sum_{q=0;\,q\text{ even}}^{p} a_{q,p} z^q = \prod_{q=1}^{p/2} (1 + (z/q)^2),^* \qquad z \in \mathbf{C}. \tag{2.3}$$

Proof. Integrating (1.19), we have

$$\int_0^h \exp\left[ir\frac{\pi}{h}x\right] dx = \sum_{q=0;\,q\text{ even}}^{p} \left(ir\frac{\pi}{h}\right)^q \int_{-\infty}^{\infty} t_q(x)\,dx, \qquad r = 0, 2, \cdots, p,$$

thus, we are led to the system of $p/2 + 1$ linear equations

$$\begin{aligned} a_{0,p} &= 1, \\ \sum_{q=0;\,q\text{ even}}^{p} (ir)^q a_{q,p} &= 0, \qquad r = 1, \cdots, p/2. \end{aligned} \tag{2.4}$$

Since the determinant D_p of (2.4) is a Vandermonde determinant with

$$D_p = i^{(p/2)(p/2+1)} \prod_{q>r=0}^{p/2} (q^2 - r^2) \neq 0,$$

the weights $a_{q,p}$ are uniquely determined by the system (2.4).

Define a polynomial P_p of degree p by

$$P_p(z) := \sum_{q=0;\,q\text{ even}}^{p} a_{q,p} z^q, \qquad z \in \mathbf{C}.$$

Then (2.4) reads

$$\begin{aligned} P_p(0) &= 1, \\ P_p(ri) &= 0, \qquad r = \pm 1, \cdots, \pm p/2. \end{aligned}$$

Hence $P_p(z) = \prod_{q=1}^{p/2} (1 + (z/q)^2)$, and (2.3) is established.

From (2.3) it follows that

$$(1 + (2z/p)^2) \sum_{q=0;\,q\text{ even}}^{p-2} a_{q,p-2} z^q = \sum_{q=0;\,q\text{ even}}^{p} a_{q,p} z^q, \qquad p = 2, 4, \cdots .$$

Comparing the coefficients, we find the desired recursion formulae

$$\begin{aligned} a_{0,p} &= 1, \qquad p = 0, 2, \cdots, \\ a_{q,p} &= a_{q,p-2} + (2/p)^2 a_{q-2,p-2}, \qquad q = 2, 4, \cdots, p-2, \quad p = 2, 4, \cdots, \\ a_{p,p} &= \frac{1}{((p/2)!)^2}, \qquad p = 0, 2, \cdots . \end{aligned} \tag{2.5}$$

Using (2.5), we obtain

* $\prod_{q=1}^{p/2}$ is to be interpreted as unity when $p = 0$.

$$I_{0,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh),$$

$$I_{2,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh) + \frac{h^3}{4\pi^2} \sum_{m=-\infty}^{\infty} f''(mh),$$

$$I_{4,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh) + \frac{5h^3}{16\pi^2} \sum_{m=-\infty}^{\infty} f''(mh) + \frac{h^5}{64\pi^4} \sum_{m=-\infty}^{\infty} f''''(mh).$$

In the case of odd p, we integrate (1.19) for $r = 0, 2, \cdots, p-1$ and get the system of $(p+1)/2$ linear equations

$$\text{(2.6)} \qquad \begin{aligned} a_{0,p} &= 1, \\ \sum_{q=0;\, q \text{ even}}^{p-1} (ir)^q a_{q,p} &= 0, \qquad r = 1, \cdots, (p-1)/2. \end{aligned}$$

Comparing (2.6) with (2.4), we see that $a_{q,p} = a_{q,p-1}$, $q = 0, 2, \cdots, p-1$. Thus, if p is odd, $I_{p,h}(f) = I_{p-1,h}(f)$ is valid. Therefore, we may restrict ourselves to even p.

In the case when the function f is analytic, we state a sufficient condition on the convergence of the series (2.1) and give a remainder in the following:

THEOREM 2.2. *Let the function f be analytic in the strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$, let $f(z) \to 0$, $z = x + iy$ as $x \to \pm\infty$ uniformly for all $-a \leqq y \leqq a$ and let*

$$\text{(2.7)} \qquad \int_{-\infty-ia}^{\infty-ia} |f(z)|\, ds < \infty, \qquad \int_{-\infty+ia}^{\infty+ia} |f(z)|\, ds < \infty.$$

Then $\int_{-\infty}^{\infty} f(x)\, dx$ exists, and the series (2.1) is convergent for even $p \geqq 0$ and $h > 0$. The remainder

$$\text{(2.8)} \qquad E_{p,h}(f) := \int_{-\infty}^{\infty} f(x)\, dx - I_{p,h}(f)$$

is given by

$$\text{(2.9)} \qquad \begin{aligned} E_{p,h}(f) &= \frac{1}{(2i)^{p+1}} \sum_{q=0}^{p/2} (-1)^q \binom{p+1}{q} \left\{ \int_{-\infty+ia}^{\infty+ia} \frac{\exp[i(p+1-2q)(\pi/h)\zeta]}{\sin^{p+1}(\pi/h)\zeta} f(\zeta)\, d\zeta \right. \\ &\qquad \left. - \int_{-\infty-ia}^{\infty-ia} \frac{\exp[-i(p+1-2q)(\pi/h)\zeta]}{\sin^{p+1}(\pi/h)\zeta} f(\zeta)\, d\zeta \right\} \end{aligned}$$

with the bound

$$\text{(2.10)} \qquad |E_{p,h}(f)| \leqq \frac{\exp[-(\pi/h)a]}{2\sinh^{p+1}(\pi/h)a} \left(\int_{-\infty+ia}^{\infty+ia} |f(z)|\, ds + \int_{-\infty-ia}^{\infty-ia} |f(z)|\, ds \right).$$

Proof. From the assumption (2.7), we see by Cauchy's theorem that $\int_{-\infty}^{\infty} f(x)\, dx$ exists.

Using the identity

$$\sin^{p+1} \frac{\pi}{h} x = \frac{\exp[i(p+1)(\pi/h)x]}{(2i)^{p+1}} \sum_{q=0}^{p+1} (-1)^q \binom{p+1}{q} \exp[-2iq(\pi/h)x],$$

we deduce from the residue theorem that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{x-\zeta}\,dx=\frac{\exp[i(p+1)(\pi/h)\zeta]}{(2i)^p}\sum_{q=0}^{p/2}(-1)^q\binom{p+1}{q}\exp[-2iq(\pi/h)\zeta],$$

$$\zeta=\xi+ia,$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{x-\zeta}\,dx=\frac{\exp[-i(p+1)(\pi/h)\zeta]}{(2i)^p}\sum_{q=0}^{p/2}(-1)^q\binom{p+1}{q}\exp[2iq(\pi/h)\zeta],$$

$$\zeta=\xi-ia.$$

From this the estimates

$$\text{(2.11)}\quad \left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{x-\zeta}\,dx\right|\leqq\frac{\exp[-(p+1)(\pi/h)a]}{2^p}\sum_{q=0}^{p/2}\binom{p+1}{q}\exp[2q(\pi/h)a]$$

$$\leqq\exp[-(\pi/h)a],\qquad \zeta=\xi\pm ia,$$

follow.

Integrating (1.8) over $(-\infty,\infty)$ and interchanging the order of integration, we obtain

$$\int_{-(n+1/2)h}^{(n+1/2)h}f(x)\,dx=h\sum_{m=-n}^{n}\sum_{q=0}^{p}\left(\frac{h}{2\pi}\right)^q a_{q,p}f^{(q)}(mh)$$

$$+\frac{1}{2\pi i}\int_{C_n}\frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta}\left(\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{\zeta-x}\,dx\right)d\zeta.$$

With the aid of (2.11), we can estimate

$$\left|\frac{1}{\pi}\int_{\pm(n+1/2)h-ia}^{\pm(n+1/2)h+ia}\frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta}\left(\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{\zeta-x}\,dx\right)d\zeta\right|$$

$$\leqq 2a\max_{\eta\in[-a,a]}|f(\pm(n+\tfrac{1}{2})h+i\eta)|,$$

and

$$\left|\frac{1}{\pi}\int_{-(n+1/2)h\pm ia}^{(n+1/2)h\pm ia}\frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta}\left(\int_{-\infty}^{\infty}\frac{\sin^{p+1}(\pi/h)x}{\zeta-x}\,dx\right)d\zeta\right|$$

$$\leqq\frac{\exp[-(\pi/h)a]}{\sinh^{p+1}(\pi/h)a}\int_{-(n+1/2)h\pm ia}^{(n+1/2)h\pm ia}|f(\zeta)|\,ds.$$

Thus, by the assumptions on f, letting $n\to\infty$ completes the proof.

Remark. From the bound (2.10), *we have*

$$\lim_{h\to 0}I_{p,h}(f)=\int_{-\infty}^{\infty}f(x)\,dx,\qquad p=0,2,\cdots,$$

and

$$\lim_{p\to\infty}I_{p,h}(f)=\int_{-\infty}^{\infty}f(x)\,dx,\qquad h>0,\ \sinh\frac{\pi}{h}a>1,$$

where convergence is exponential in both cases $h\to 0$, fixed and $p\to\infty$, h fixed.

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