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High Precision Evaluation of the Irregular Coulomb Wave Functions

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Abstract. This tutorial paper presents practical methods for accurately evaluating irregular Coulomb wave functions. Rational approximations to $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$ are developed along line segments in the (η, ρ) plane to provide useful initial values for the associated differential equation. These approximations, designed for IBM System 360 Fortran double precision, yield results to at least 13 significant decimal places.

1. **Introduction.** Considerable interest in Coulomb wave functions developed from physical problems involving collisions of charged particles. These functions $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, called the regular and irregular Coulomb wave functions, are two linearly independent solutions of the second-order differential equation

$$d^2y/d\rho^2 + [1 - 2\eta/\rho - L(L + 1)/\rho^2]y = 0 \tag{1.1}$$

for real η, positive ρ, and nonnegative integer L.

The regular Coulomb wave functions are readily evaluated using well-known methods [3], [9]. A useful three-term recurrence relation technique for generating $F_L(\eta, \rho)$, working from higher L values to lower ones, is discussed in detail by Gautschi [10], [11]. Unfortunately, because of the different asymptotic properties with respect to L between the regular and irregular Coulomb wave functions, this technique cannot be safely applied to the irregular functions. A generation from lower to higher L values using the recurrence relation is possible, but then the exact numerical values of two consecutive $G_L(\eta, \rho)$ functions are required.

Because irregular Coulomb wave functions are difficult to compute, current tables [2], [3], [8], [17], [19], [23] contain relatively few digits. As examples, taken from Luke [16], which indicate computational difficulties that can be encountered, consider the related equations

$$F_L(\eta, \rho) + iG_L(\eta, \rho) = \frac{(-1)^{L+1}\Gamma(L + 1 - i\eta)e^{\pi\eta/2}(2\rho)^{L+1}e^{-i\rho}}{|\Gamma(L + 1 - i\eta)|}\Psi(L + 1 - i\eta; 2L + 2; 2i\rho)$$

and

$$F_L(\eta, \rho) + iG_L(\eta, \rho) = \frac{\Gamma(L + 1 - i\eta)e^{\pi\eta/2}i^{L+1}}{|\Gamma(L + 1 - i\eta)|} W_{i\eta, L+1/2}(2i\rho),$$

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where

$$\Psi(a; c; z) = \int_0^\infty e^{-zt} t^{a-1}(1 + t)^{c-a-1}\, dt/\Gamma(a), \qquad R(a) > 0, \quad R(z) > 0,$$

is the confluent hypergeometric function and $W_{k,m}(z)$ is the Whittaker confluent hypergeometric function. In fact, a method for computing irregular Coulomb wave functions using Whittaker functions was recently developed [5], and found to work fairly well for small parameters, but was subject to severe cancellations for larger ones [20]. Many well-known formulas are either too specialized, too complicated, or not sufficiently accurate to be immediately useful for the uninitiated user. Furthermore, errors in earlier works are noted in more recent literature [9], [13]. Some of these difficulties arise from attempts to represent $G_L(\eta, \rho)$ directly in terms of all three parameters simultaneously.

In this report, we develop accurate but relatively simple approximations for $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$ along certain line segments in the positive region of the (η, ρ) plane, eliminating the complications of three-dimensional representations. The coefficients for these approximation formulas are given in tables on the microfiche card attached to this issue.

2. Basic Relations. Some of the formulas on which this work is based were previously given by Abramowitz [3], Fröberg [9], and others. The most useful of these are summarized below.

$$(2.1) \qquad (1 + \eta^2)^{1/2} \cdot \Gamma_1 = (\rho^{-1} + \eta) \cdot \Gamma_0 - \Gamma_0',$$

$$(2.2) \qquad L[(L + 1)^2 + \eta^2]^{1/2} \cdot \Gamma_{L+1} = (2L + 1)[\eta + L(L + 1)/\rho] \cdot \Gamma_L - (L + 1)(L^2 + \eta^2)^{1/2} \cdot \Gamma_{L-1},$$

where Γ_L can be taken as $F_L(\eta, \rho)$ or $G_L(\eta, \rho)$.

$$(2.3) \qquad F_L + iG_L = \frac{e^{-\pi\eta}\rho^{L+1}}{(2L + 1)!\, C_L(\eta)} \int_0^\infty \{(1 - \tanh^2 t)^{L+1} \exp[-i(\rho \tanh t - 2\eta t)] + i(1 + t^2)^L \exp[-\rho t + 2\eta \arctan t]\}\, dt,$$

where

$$C_0(\eta) = (2\pi\eta(e^{2\pi\eta} - 1)^{-1})^{1/2} \quad \text{and} \quad C_L(\eta) = \frac{(L^2 + \eta^2)^{1/2}}{L(2L + 1)} C_{L-1}(\eta).$$

$$(2.4) \qquad \begin{matrix} F_0(\eta, \rho) \\ G_0(\eta, \rho) \end{matrix} \simeq \pi^{1/2}(2\eta)^{1/6} \left\{ \begin{matrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \left[1 + \frac{g_1}{\mu} + \frac{g_2}{\mu^2} + \cdots \right] + \begin{matrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{matrix} \left[\frac{f_1}{\mu} + \frac{f_2}{\mu^2} + \cdots \right] \right\},$$

$$(2.5) \qquad \begin{matrix} F_0'(\eta, \rho) \\ G_0'(\eta, \rho) \end{matrix} \simeq -\pi^{1/2}(2\eta)^{-1/6} \left\{ \begin{matrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \left[\frac{g_1' + xf_1}{\mu} + \frac{g_2' + xf_2}{\mu^2} + \cdots \right] + \begin{matrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{matrix} \left[1 + \frac{(g_1 + f_1')}{\mu} + \frac{(g_2 + f_2')}{\mu^2} + \cdots \right] \right\},$$

where

$$\begin{aligned} f_1 &= x^2/5, \\ f_2 &= (2x^3 + 6)/35, \\ f_3 &= (84x^7 + 1480x^4 + 2320x)/63000, \\ g_1 &= -x/5, \\ g_2 &= (7x^5 - 30x^2)/350. \\ g_3 &= (1056x^6 - 1160x^3 - 2240)/63000, \\ x &= (2\eta - \rho)/(2\eta)^{1/3}, \\ \mu &= (2\eta)^{2/3}, \qquad \eta \gg 0,\ |\rho - 2\eta| < 2\eta, \end{aligned}$$

and Ai(x) and Bi(x) are the Airy functions. For further details, see [3], [9], [21], [25].

$$F_L'(\eta, \rho)\cdot G_L(\eta, \rho) - F_L(\eta, \rho)\cdot G_L'(\eta, \rho) = 1. \tag{2.6}$$

$$F_{L-1}(\eta, \rho)\cdot G_L(\eta, \rho) - F_L(\eta, \rho)\cdot G_{L-1}(\eta, \rho) = L(L^2 + \eta^2)^{-1/2}. \tag{2.7}$$

In addition, the Leibniz rule for high order derivatives of products

$$\begin{aligned} (\alpha\cdot\beta)^{(k)} &= \alpha^{(k)}\cdot\beta + \binom{k}{1}\alpha^{(k-1)}\cdot\beta^{(1)} + \binom{k}{2}\alpha^{(k-2)}\cdot\beta^{(2)} \\ &\quad + \cdots + \binom{k}{1}\alpha^{(1)}\cdot\beta^{(k-1)} + \alpha\cdot\beta^{(k)} \end{aligned} \tag{2.8}$$

is especially useful. Note that any one of the functions $\alpha\cdot\beta$, α, or β and its derivatives can be calculated in the proper sequence whenever the other two and their derivatives are known numerically.

3. Generation of Initial Values. We generally use two types of formulas. The first, applicable for reasonably small η and ρ values, consists of the integral representation (2.3) for $L = 0$ and 1; the second, applicable for very large η, consists of the Airy function expansions (2.4) and (2.5).

Because the integral (2.3) is relatively difficult to evaluate with arbitrarily high precision, we recommend the procedure outlined below.

Each term in $F_L(\eta, \rho)$ or $G_L(\eta, \rho)$ contains an integral $\int_0^\infty \alpha(t)\beta(t)\,dt$ which is safely truncated at a suitably large t. The significant portion of this integral is evaluated to a specified degree of precision by a sum of integrals of the type $I(h) = \int_x^{x+h} \alpha(t)\beta(t)\,dt$. $I(h)$, when expanded into the Taylor series

$$I(h) \simeq \sum_{k=1}^{N} h^k[\alpha(x)\beta(x)]^{(k-1)}/k!,$$

is easily computed from (2.8) when α, β, and their derivatives are known numerically. For a given N (the authors use $N = 54$ with the CDC 3600 computer), a suitably large h value is chosen, dependent on the magnitudes of the coefficients of the highest order terms used by the Taylor series, so that the truncation terms remain negligible.

If $T = 1 - \tanh^2 x = \operatorname{sech}^2 x = 4z^{-1}$, then $z = e^{2x} + e^{-2x} + 2$, $z^{(k)} =$

$2^k(e^{2x} + (-1)^k e^{-2x})$ and $(zT)^{(k)} = 0$, $k > 0$. Thus, from (2.8), $T^{(k)}$ is obtained in terms of lower order derivatives of T and z.

If $A = \rho \tanh x - 2\eta x$, $C = \cos A$ and $S = \sin A$, then $A^{(1)} = \rho T - 2\eta$, $A^{(1+k)} = \rho T^{(k)}$, $S^{(1)} = CA^{(1)}$ and $C^{(1)} = -SA^{(1)}$. $S^{(1+k)}$ and $C^{(1+k)}$ are available in the proper sequence from Leibniz's rule applied to $S^{(1)}$ and $C^{(1)}$.

If $v = -\rho x + 2\eta \arctan x$, the higher derivatives of v are computed from (2.8) applied to the equation $(1 + x^2)(v^{(1)} + \rho) = 2\eta$. When $B = e^v$, $B^{(1)} = Bv^{(1)}$, and desired results follow naturally as before.

This technique obtained accurate values from which the more efficient approximation formulas were developed. It was also used to check numerical results obtained by other methods. It took approximately four minutes of CDC 3600 computer time to calculate $F_L(11, 22)$ and $G_L(11, 22)$ for $L = 0$ and 1 to 22 decimal digit accuracy. The efficiency of this method improves for smaller η and ρ values.

The value $G_0'(\eta, \rho)$ is determined by $G_0(\eta, \rho)$ and $G_1(\eta, \rho)$ according to Eq. (2.1).

4. Approximations for $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$ Along Line Segments. By rearranging the Airy expansions (2.4) and (2.5) into continued fractions along the transition line $\rho = 2\eta$, we obtain

$$G_0(\eta, 2\eta) \simeq \lambda^{-1/12} \cdot G + \lambda^{7/12} \cdot H \tag{4.1}$$

and

$$G_0'(\eta, 2\eta) \simeq \lambda^{1/12} \cdot G' + \lambda^{5/12} \cdot H', \tag{4.2}$$

where $\lambda = (2\eta)^{-2}$ and G, H, G', and H' are of the form $G = G_1/1 - \lambda G_2/1 - \lambda G_3/1 - \cdots$ with the coefficients listed in Table 1. We can also derive these same expansions from the asymptotic series [9], [13]. Test calculations indicated that these continued fraction approximations are accurate over a much larger range than the corresponding Airy expansions from which they were derived.

Rational minimax approximations based on the Remes algorithm [6], [24] were ultimately obtained along the transition line for $1 < \eta \leqq 15$ from the integration technique described in the preceding section, using intermediate Chebyshev polynomial approximations to speed the required computations. Coefficients for these approximations, with ranges $3.5 < \eta \leqq 15$, $2 < \eta \leqq 3.5$, and $1 \leqq \eta \leqq 2$, are given in Tables 2, 3, and 4, respectively.

In Table 5, we give coefficients for rational approximations along the line $\rho = 1$ for $\eta < 1$.

We also applied Eq. (1.1) to obtain numerical values for $G_0(\eta, 30)$ and $G_0'(\eta, 30)$. In Tables 6 and 7, we have coefficients for rational approximations to these two functions for the intervals $15 \leqq \eta \leqq 18.5$ and $18.5 < \eta \leqq 22$, and in Table 8, we list coefficients for approximations to $\ln(G_0(\eta, 30))$ and $\ln(-G_0'(\eta, 30))$ for $22 < \eta \leqq 30$. These rational approximations at $\rho = 30$ are given in terms of the Chebyshev polynomials $T_0(x) = 1$, $T_1(x) = x$, and $T_m(x) = 2x \cdot T_{m-1}(x) - T_{m-2}(x)$, $m > 1$, because the direct polynomial representations lead to cancellation errors.

Some relevant numerical values of these functions are provided in Table 9 for checking purposes. Test cases lead us to believe all digits in this table are significant, with possible roundoff errors in the last place.

5. Calculation of $G_L(\eta, \rho)$. If the point (η, ρ) is not on one of the line segments where our approximation formulas hold, then Eq. (1.1) determines both $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$ at the desired point, assuming the initial values obtained from these rational approximations.

Since (1.1) implies

$$\rho y^{(k+1)} = (2\eta - \rho)y^{(k-1)} - (k - 1)(y^{(k-2)} + y^{(k)}) \quad \text{with } k \geqq 1,$$

the higher order derivatives are conveniently applied to Taylor series expansions for both $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$. When the step size is varied and maximized according to magnitudes of high order terms in the series, the desired results are derived accurately in a relatively small number of steps. However, it may be necessary in certain regions to compute scaled derivatives $s^k \cdot y^{(k)}/k!$ to keep intermediate values in the range of computer number systems.

Once $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$ are known numerically, the consecutive $G_L(\eta, \rho)$ values for $L > 0$ follow from Eqs. (2.1) and (2.2).

6. Checking of Results. The regular Coulomb functions were generally computed with the irregular functions for $L = 0$ and 1 because common terms often appear in the mathematical formulas and because it is desirable to use the Wronskians (2.6) and (2.7) to check on obvious errors. Unfortunately, these Wronskians proved to be very poor checks on the validity of final results. The asymptotic, Airy, and Riccati methods described by Fröberg [9] show that the regular and irregular Coulomb wave functions are all based on common terms which are cancelled out completely in the Wronskians. In the asymptotic method, θ_L could be miscalculated by several orders of magnitude and the Wronskians would fail to show any inaccuracies.

The Airy function approximations (2.4) and (2.5) for $F_0(\eta, \rho)$, $F_0'(\eta, \rho)$, $G_0(\eta, \rho)$ and $G_0'(\eta, \rho)$, expanded to μ^{-36}, were used to obtain initial values which were applied to the differential equation (1.1) to obtain these four functions at another point (η, ρ_1). No significant error was detected by recomputing these four values from the Airy expansions at (η, ρ_1). The initial values used in these tests were taken along the transition line $\rho = 2\eta$, where these Airy approximations are equivalent to the asymptotic series [1], [9], [13], [14], [15], [17]. For $11 \leqq \eta < 15$, numerical results from the Airy expansions were found to be in very good agreement with those obtained from the integral expressions (2.3).

For $\eta \leqq 50$, $\rho > 2\eta + 33$, numerical results from (4.1), (4.2), and (1.1) were compared to values obtained by the asymptotic method [9]. The function $\arg \Gamma(1 + i\eta)$ required in the expression for θ_L was computed using a subroutine developed by Cody and Hillstrom [7].

For $\eta > 50$, values obtained from two types of Riccati methods [9] were compared to results from (4.1), (4.2), and (1.1).

As formulas were verified, several CDC 3600 computer subroutines were designed to produce master values to at least 21 decimals. Later, a Fortran double precision subroutine for calculating $G_L(\eta, \rho)$ for $L = 0, 1, \cdots, n$ was designed for IBM 360 computers. This 360 subroutine, used on Argonne's 50–75 computer, calculated results for $L = 0$ and 1 which were found to be accurate to at least 13S for $\rho < 2\eta$ and to at least 13D when $\rho > 2\eta$ whenever $\rho > .2$. A test program using this subroutine took less than 24 seconds to obtain all 800 values of $G_0(\eta, \rho)$ and all 800 values of $G_0'(\eta, \rho)$ which appear in the U. S. National Bureau of Standards Handbook of Mathematical Functions [3].

7. Summary. The main purposes of this report are (1) to provide useful approximations to initial values of the commonly used differential equation, (2) to show how complicated integrals such as (2.3) can be numerically evaluated in a relatively efficient manner with high precision, and (3) to suggest methods which make it possible to obtain much more accurate results if they are ever required in the future.

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TABLES FOR APPROXIMATING

IRREGULAR COULOMB WAVE FUNCTIONS:

$G_o(\eta, \rho)$ and $G_o'(\eta, \rho)$

by

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TABLE 1

MAXIMUM RELATIVE ERROR < 10^{-21}

COEFFICIENTS FOR EVALUATING $G_0(\eta,2\eta)$, $\eta > 15$, FROM EQUATION (4.1)

m	G_m	H_m
1	1.08992 90688 41005 58588 58452 (0)	1.36212 07290 99253 08471 88544 (-1)
2	-3.55555 55555 55555 55555 55555 (-2)	1.98017 09401 70940 17094 01792 (-1)
3	4.45458 66831 58111 72954 02790 (-1)	1.17924 30590 68257 42326 12045 (0)
4	1.27829 18410 79505 28166 46765 (0)	2.32422 57255 23372 13886 33179 (0)
5	3.02498 45091 26205 01314 65319 (0)	4.38157 64843 97042 15054 41413 (0)
6	4.67798 59675 21372 71630 05527 (0)	6.45171 90006 22568 61890 17615 (0)
7	7.60100 07216 46544 74981 80208 (0)	9.58198 61188 29209 94013 94604 (0)
8	1.00882 12638 08937 02993 43187 (1)	1.25823 72050 03530 28121 58607 (1)
9	1.41763 73899 55079 06648 34397 (1)	1.67810 07483 48622 97606 64719 (1)
10	1.75036 56765 31090 30227 97301 (1)	2.07151 05731 27222 73131 17614 (1)
11	2.27510 32043 39371 15618 87626 (1)	2.59789 47155 43234 15654 47607 (1)
12	2.69223 24448 82441 36545 50635 (1)	3.08493 21291 17398 52539 88644 (1)
13	3.33249 74997 78933 00037 42696 (1)	

COEFFICIENTS FOR EVALUATING $G_0'(\eta,2\eta)$, $\eta > 15$, FROM EQUATION (4.2)

m	G'_m	H'_m
1	-7.94570 42530 78976 32752 66512 (-1)	2.17985 31376 82011 17377 16906 (-1)
2	1.26984 12698 41269 84126 98491 (-3)	8.28860 02886 00288 60028 85851 (-2)
3	3.92718 92551 89255 18925 50842 (0)	9.30621 06753 73382 23885 75185 (-1)
4	-1.19800 54929 97099 35164 98399 (0)	1.88022 36632 26213 01189 18073 (0)
5	-4.82654 49062 33677 53525 57485 (0)	3.84795 41457 88628 45817 95486 (0)
6	1.38112 08828 93547 26720 03171 (1)	5.66863 97065 97011 09437 11343 (0)
7	3.06062 11591 29591 60569 30630 (0)	8.75273 11882 85234 19849 22527 (0)
8	1.61097 59315 93691 55735 77536 (1)	1.14569 42091 45603 79945 58516 (1)
9	9.80663 19460 37811 33619 12922 (0)	1.56514 62995 37258 33327 00837 (1)
10	2.35160 17189 77909 38160 59734 (1)	1.92456 63615 44725 10369 78027 (1)
11	1.80839 82624 30568 59741 36355 (1)	2.45463 45932 82860 13534 69098 (1)
12	3.33715 75822 80725 96042 71018 (1)	2.90348 76009 55463 55998 00631 (1)
13	2.81790 23555 18866 41495 62182 (1)	

TABLE 2

MAXIMUM RELATIVE ERROR < $2\cdot 10^{-14}$

$$\omega = (2\eta)^{2/3}, \quad 3.5 < \eta \le 15$$

$$G_0(\eta,2\eta) \approx \omega^{1/4} \cdot \sum_{m=1}^{9} P_m \omega^{m-1} \Big/ \sum_{m=1}^{9} Q_m \omega^{m-1}$$

m	P_m	Q_m
1	2.23451 87399 01281 25630 (1)	2.95775 68741 29461 18866 (1)
2	-1.13233 76817 31477 15736 (2)	-9.20215 03059 40531 00603 (1)
3	5.78246 04336 08344 22429 (0)	-3.16762 08809 89725 33079 (1)
4	-7.93227 78854 20256 78925 (1)	-4.43020 29315 92870 63573 (1)
5	2.72907 83220 80819 16464 (2)	2.40254 08979 32696 33255 (2)
6	-2.26577 27720 44742 80414 (2)	-2.06113 44519 05309 52418 (2)
7	8.42380 57501 61425 26230 (1)	7.71626 77996 01883 12916 (1)
8	-1.51192 03892 84148 51391 (1)	-1.38717 31962 86043 76397 (1)
9	1.08992 90687 42122 66690 (0)	1.00000 00000 00000 00000 (0)

$$G_0'(\eta,2\eta) \approx \omega^{-1/4} \cdot \sum_{m=1}^{9} P_m' \omega^{m-1} \Big/ \sum_{m=1}^{7} Q_m' \omega^{m-1}$$

m	P_m'	Q_m'
1	-4.29906 33488 05974 36388 (-1)	1.26983 59143 93520 03600 (0)
2	-2.33530 26164 95128 82249 (0)	9.87595 98634 18596 84462 (-1)
3	3.00589 76366 56352 90426 (0)	-6.38075 40945 51418 74382 (0)
4	1.13118 99965 62795 75005 (1)	-9.59493 87670 57619 39435 (0)
5	-1.50181 80641 79841 35219 (1)	1.69246 38860 41678 84779 (1)
6	5.94213 93227 42431 39469 (0)	-7.20408 79425 90229 72050 (0)
7	-7.94570 34892 57121 75194 (-1)	1.00000 00000 00000 00000 (0)
8	-2.05974 13357 52840 90264 (-9)	
9	2.53663 88048 25975 52407 (-11)	

TABLE 3

MAXIMUM RELATIVE ERROR < 10^{-14}

$$\omega = (2\eta)^{2/3}, \quad 2 < \eta \leq 3.5$$

$$G_0(\eta,2\eta) = \omega^{1/4} \cdot \sum_{m=1}^{8} P_m \omega^{m-1} \Big/ \sum_{m=1}^{8} Q_m \omega^{m-1}$$

m	P_m	Q_m
1	1.24967 47443 67499 79414 (0)	5.16802 66162 77380 80042 (0)
2	-1.45115 87977 23446 36893 (1)	-2.23522 31831 89546 94991 (1)
3	-5.61040 30939 66625 76074 (0)	3.34736 92818 42808 31805 (0)
4	5.15035 99603 69915 64772 (1)	4.30394 62995 61734 62207 (1)
5	-6.02124 65030 45280 77235 (1)	-5.41410 75129 80351 59335 (1)
6	3.28708 08039 82716 03318 (1)	3.00307 73869 69386 39214 (1)
7	-9.07509 20740 11020 04583 (0)	-8.32612 94944 32240 27052 (0)
8	1.08993 47919 85988 63627 (0)	1.00000 00000 00000 00000 (0)

$$G_0'(\eta,2\eta) = \omega^{-1/4} \cdot \sum_{m=1}^{8} P_m' \omega^{m-1} \Big/ \sum_{m=1}^{8} Q_m' \omega^{m-1}$$

m	P_m'	Q_m'
1	1.14748 51278 16229 54634 (1)	-1.34924 50836 45477 54603 (0)
2	-6.61279 99071 20111 05493 (1)	4.94389 33529 34736 27971 (1)
3	1.31172 21218 58400 04652 (2)	-1.24032 34742 64506 98743 (2)
4	-1.41223 59211 21115 82064 (2)	1.49796 53253 50803 12559 (2)
5	8.99184 40244 41465 89046 (1)	-1.01817 87387 67506 47431 (2)
6	-3.49140 24952 98328 27090 (1)	4.13519 06869 99234 15885 (1)
7	7.71494 27545 95630 50299 (0)	-9.43521 95509 43055 91410 (0)
8	-7.94570 54814 15770 38806 (-1)	1.00000 00000 00000 00000 (0)

TABLE 4

MAXIMUM RELATIVE ERROR < 10^{-14}

$$\omega = (2\eta)^{2/3},\ 1 < \eta \le 2$$

$$G_0(\eta,2\eta) = \omega^{1/4} \cdot \sum_{m=1}^{8} P_m \omega^{m-1} \Big/ \sum_{m=1}^{8} Q_m \omega^{m-1}$$

m	P_m		Q_m	
1	4.98275 65752 20298 31180	(0)	2.22504 12276 34888 16737	(0)
2	1.54146 07709 13096 42225	(1)	1.62301 90569 76947 51874	(1)
3	-3.06880 33760 40014 27341	(1)	-2.90616 84270 29017 84857	(1)
4	4.51275 59174 91839 39642	(1)	4.12481 79971 39485 16345	(1)
5	-3.41019 01260 24645 94786	(1)	-3.11642 78288 44479 68789	(1)
6	1.89619 13241 60115 95479	(1)	1.73671 38319 90676 45593	(1)
7	-5.79158 33481 57625 28869	(0)	-5.32266 25776 79406 93394	(0)
8	1.08952 43901 12571 45947	(0)	1.00000 00000 00000 00000	(0)

$$G_0'(\eta,2\eta) = \omega^{-1/4} \cdot \sum_{m=1}^{8} P'_m \omega^{m-1} \Big/ \sum_{m=1}^{8} Q'_m \omega^{m-1}$$

m	P'_m		Q'_m	
1	9.22650 57356 51310 94552	(-1)	6.66787 16651 42472 60872	(0)
2	-1.06345 60384 16184 15236	(1)	3.32206 35987 17696 36702	(0)
3	1.19338 77167 47101 98218	(1)	-3.60205 52360 82315 35552	(0)
4	-2.03631 25690 41060 82555	(1)	1.81504 94433 27106 69539	(1)
5	1.70174 83350 68204 48881	(1)	-1.71513 81505 52457 70396	(1)
6	-1.12316 25080 55069 18563	(1)	1.27889 95380 74053 05943	(1)
7	3.75667 86924 46009 51257	(0)	-4.44122 84787 08240 67551	(0)
8	-7.94986 79900 12537 23498	(-1)	1.00000 00000 00000 00000	(0)

TABLE 5

MAXIMUM ABSOLUTE ERROR < 10^{-16}

$$G_0(\eta,1) = \sum_{m=1}^{11} P_m \eta^{m-1} \Big/ \sum_{m=1}^{11} Q_m \eta^{m-1}, \quad 0 < \eta \leq 1$$

m	P_m	Q_m
1	2.39216 08886 28766 14170 (3)	4.42744 89718 95690 58453 (3)
2	4.94441 36244 57384 11649 (3)	-2.37172 39367 52091 03159 (3)
3	1.21340 17512 83979 86148 (3)	9.63880 89864 44499 25803 (3)
4	9.28481 56556 92361 41624 (3)	-5.35810 24596 20256 49438 (3)
5	-2.02925 88737 25929 66971 (3)	7.07140 18822 69215 10305 (3)
6	5.84594 77374 05361 99331 (3)	-3.86731 07444 46791 17982 (3)
7	-8.98301 87616 20064 19190 (2)	2.05421 88548 54012 52930 (3)
8	1.26038 75506 61042 89148 (3)	-9.05585 76941 81329 70747 (2)
9	1.98988 71778 31432 96382 (2)	2.19124 81138 67837 37402 (2)
10	-2.17916 76465 03823 75014 (1)	-2.49398 75694 14483 55023 (1)
11	4.70021 38718 08131 96441 (1)	1.00000 00000 00000 00000 (0)

$$G_0'(\eta,1) = \sum_{m=1}^{11} P'_m \eta^{m-1} \Big/ \sum_{m=1}^{11} Q'_m \eta^{m-1}, \quad 0 < \eta \leq 1$$

m	P'_m	Q'_m
1	-1.19341 30147 48728 44860 (3)	1.42418 81614 28192 29102 (3)
2	2.02176 52257 37445 07429 (3)	-8.04859 03063 89477 30685 (2)
3	-3.57969 79922 59092 39988 (3)	3.28876 01955 38269 10516 (3)
4	2.75036 00562 37287 14649 (3)	-2.09606 66775 44305 20041 (3)
5	-3.47127 15457 35339 42633 (3)	2.66132 50808 96058 10885 (3)
6	4.20899 77897 59953 61263 (2)	-1.73444 47266 64645 71860 (3)
7	-7.24986 21227 02665 43680 (2)	9.37785 94080 81895 51105 (2)
8	-5.66938 74508 04113 41944 (2)	-4.71102 43325 90178 51880 (2)
9	3.07098 13357 34991 91783 (2)	1.43855 30131 15173 16081 (2)
10	-1.98184 39857 63508 92380 (2)	-2.08554 33866 39866 80220 (1)
11	3.21116 49035 77880 20691 (1)	1.00000 00000 00000 00000 (0)

TABLE 6

MAXIMUM RELATIVE ERROR < $2 \cdot 10^{-15}$

$x = (4n-67)/7, \; 15 < n \le 18.5$

$$G_0(n,30) = (.5Q_0 + \sum_{m=1}^{10} Q_m T_m(x)) / (.5R_0 + \sum_{m=1}^{8} R_m T_m(x))$$

m	Q_m	R_m
0	2.13913 91796 65545 93479 (-2)	4.27075 04804 54062 21862 (-3)
1	6.94044 20967 03597 65104 (-3)	-5.76663 28270 21477 77676 (-4)
2	1.84519 39925 08152 90269 (-3)	-2.81102 65516 06774 15590 (-5)
3	4.65914 42991 70709 81986 (-4)	-1.01467 28510 90191 91038 (-6)
4	3.88815 21669 17106 89286 (-5)	1.28113 37033 91165 73605 (-6)
5	6.40346 06026 12869 10764 (-6)	-2.83236 88979 60446 88386 (-8)
6	6.19010 23225 20815 61986 (-7)	-4.54981 45516 83128 37280 (-9)
7	-5.58521 13234 38620 36410 (-8)	-6.97083 16106 89087 08668 (-10)
8	-1.96437 99225 11009 35014 (-10)	7.49351 24310 05741 81099 (-11)
9	-4.96632 45606 54905 65235 (-10)	
10	-1.13301 42240 52665 91491 (-10)	

$$G_0'(n,30) = (.5Q_0' + \sum_{m=1}^{10} Q_m' T_m(x)) / (.5R_0' + \sum_{m=1}^{8} R_m' T_m(x))$$

m	Q_m'	R_m'
0	-7.80064 47591 64970 87872 (-3)	4.50608 77477 30656 47684 (-3)
1	-4.01172 03263 04579 81707 (-3)	-4.68781 59749 90105 76537 (-4)
2	-1.55402 71017 06791 56891 (-3)	-1.25580 51756 27117 20785 (-5)
3	-3.08822 09964 28529 33617 (-4)	-4.77172 39029 11075 10612 (-6)
4	-6.49712 64752 06552 84278 (-5)	1.08167 81717 95120 36396 (-6)
5	-1.05303 72060 62761 48862 (-5)	-2.13819 80087 89206 56706 (-8)
6	-9.87323 57700 98102 41405 (-7)	7.89800 51531 72483 91099 (-10)
7	-1.45423 83395 43903 21995 (-7)	-8.19994 56773 53688 48533 (-10)
8	-1.11595 30300 39570 30245 (-8)	5.46455 97947 00012 67759 (-11)
9	-4.45570 89915 37746 31562 (-10)	
10	-7.35997 60842 19125 19372 (-11)	

TABLE 7

MAXIMUM RELATIVE ERROR < $5\cdot 10^{-14}$

$x = (4\eta-81)/7,\ 18.5 < \eta \le 22$

$$G_0(\eta,30) = (.5Q_0 + \sum_{m=1}^{8} Q_m T_m(x))/(.5R_0 + \sum_{m=1}^{8} R_m T_m(x))$$

m	Q_m	R_m
0	3.06242 89398 52701 49978 (-2)	4.15337 76467 86082 02236 (-4)
1	1.31265 39630 36078 84329 (-2)	-1.55092 93996 92135 96558 (-4)
2	3.36213 07188 54008 66024 (-3)	2.26915 72151 91269 78189 (-5)
3	5.61435 30115 83743 15263 (-4)	-1.18776 28748 04064 99481 (-6)
4	6.90819 17755 01534 64041 (-5)	-4.14857 90563 73524 16471 (-9)
5	5.97220 18028 30658 79093 (-6)	-9.95286 71740 92241 85709 (-9)
6	3.50579 26612 26254 50136 (-7)	2.75590 01153 77538 12745 (-9)
7	1.25672 63948 43320 34830 (-8)	-2.42029 29918 51702 99149 (-10)
8	-3.45949 71199 66232 59836 (-10)	7.85695 57415 39120 31149 (-12)

$$G_0'(\eta,30) = (.5Q_0' + \sum_{m=1}^{9} Q_m' T_m(x))/(.5R_0' + \sum_{m=1}^{8} R_m' T_m(x))$$

m	Q_m'	R_m'
0	-4.00180 47839 79352 04051 (-2)	8.22317 41736 13641 60038 (-4)
1	-2.13812 66538 55442 51854 (-2)	-2.69477 06440 53956 57675 (-4)
2	-6.53198 84598 55910 08404 (-3)	3.19023 41493 15426 11615 (-5)
3	-1.37622 65697 94629 34096 (-3)	-9.17557 25135 23969 95095 (-7)
4	-2.15570 59318 20970 23078 (-4)	-4.05844 12649 33571 80365 (-8)
5	-2.60160 76348 73606 58054 (-5)	-1.63951 73414 25255 92681 (-3)
6	-2.46948 12881 45985 40261 (-6)	3.47087 93163 87129 36487 (-9)
7	-1.76699 90170 72987 70340 (-7)	-2.40800 80843 82430 41331 (-10)
8	-9.45998 63011 78703 59293 (-9)	6.11819 49754 37915 76092 (-12)
9	-3.04542 47394 23238 90854 (-10)	

TABLE 8

MAXIMUM RELATIVE ERROR < $5 \cdot 10^{-14}$

$x = (n-26)/4, \; 22 < n \leq 30$

$$\ln(G_0(n,30)) = (.5Q_0 + \sum_{m=1}^{9} Q_m T_m(x))/(.5R_0 + \sum_{m=1}^{9} R_m T_m(x))$$

m	Q_m	R_m
0	4.56619 62284 14787 68258 (13)	3.42284 36911 90195 98104 (12)
1	2.19416 31205 27288 65479 (13)	1.20817 31279 15808 30264 (12)
2	-3.50633 53280 04029 37229 (12)	-5.03163 72261 82393 39673 (11)
3	-7.60759 36080 16101 04137 (12)	-5.10860 32307 16331 25388 (11)
4	-3.52515 22591 35876 49928 (12)	-1.72812 66953 79771 25051 (11)
5	-8.80709 70508 12006 75083 (11)	-3.05472 37048 14993 56654 (10)
6	-1.31323 37399 66247 03307 (11)	-2.84648 92894 96720 34161 (9)
7	-1.14233 24089 42889 94387 (10)	-1.21519 54372 68599 17880 (8)
8	-5.13519 77005 76763 31345 (8)	-1.65037 48735 49433 65810 (6)
9	-8.48217 86737 73511 33525 (6)	1.02400 00000 00000 00000 (3)

$$\ln(-G_0'(n,30)) = (.5Q_0' + \sum_{m=1}^{9} Q_m' T_m(x))/(.5R_0' + \sum_{m=1}^{9} R_m' T_m(x))$$

m	Q_m'	R_m'
0	-4.79319 44872 80681 13161 (14)	-3.11379 93282 97071 44771 (13)
1	-3.70270 63032 29876 06464 (14)	-2.24023 89913 86120 38554 (13)
2	-1.82565 93265 10993 01615 (14)	-9.63377 39762 20337 99190 (12)
3	-6.07643 52037 45037 17692 (13)	-2.66923 64335 75192 73805 (12)
4	-1.40331 83927 14293 74694 (13)	-4.88097 80199 07184 73316 (11)
5	-2.25220 83902 90167 14294 (12)	-5.77664 71303 41195 46908 (10)
6	-2.44789 47416 97026 70028 (11)	-4.12040 23302 74833 36625 (9)
7	-1.68914 32337 20731 94082 (10)	-1.48832 80620 78864 53595 (8)
8	-6.42956 53329 71113 05383 (8)	-1.82649 32291 09100 31929 (6)
9	-9.46498 13638 74980 95966 (6)	1.02400 00000 00000 00000 (3)

TABLE 9

η	ρ	$G_0(\eta,\rho)$		$G_0'(\eta,\rho)$	
.25	.5	1.1482084 8696547 4130711	(0)	-4.8029212 2691679 9319881	(-1)
.5	1	1.1974869 7079848 5546289	(0)	-5.6132351 5538948 3526333	(-1)
.75	1.5	1.2379327 3893294 8680233	(0)	-5.8079680 4239306 0855019	(-1)
1	1	2.0430971 6210353 8072355	(0)	-1.2635981 1331245 2681395	(0)
1	2	1.2757787 8476827 6589182	(0)	-5.8272881 3097184 7433608	(-1)
1	3	6.2703951 4889285 5262170	(-1)	-7.4782919 4000704 9735012	(-1)
1	4	-1.8900822 2659412 2104639	(-1)	-8.3273190 3385418 8694490	(-1)
1	5	-8.9841435 9092020 5486929	(-1)	-5.1080475 8519035 0105846	(-1)
1	100	9.9266047 2445652 0486889	(-1)	-1.5587643 8713855 9698599	(-1)
1.25	2.5	1.3106040 8522260 7840692	(0)	-5.7905907 8471091 1819025	(-1)
1.5	3	1.3422905 6073484 6743433	(0)	-5.7358022 0118165 1084588	(-1)
1.75	3.5	1.3711211 9481044 1859582	(0)	-5.6751874 1047685 9573730	(-1)
2	1	9.8003357 6844524 6642043	(0)	-1.3812624 1208365 8895796	(1)
2	2	3.5123830 9130932 8645479	(0)	-2.5554284 1067469 9965563	(0)
2	4	1.3974834 2679301 5884270	(0)	-5.6167098 7933261 9703058	(-1)
2.5	5	1.4442027 2109402 9777042	(0)	-5.5045577 4990027 6716261	(-1)
3	6	1.4847003 0304955 3169057	(0)	-5.4037391 4869506 7078134	(-1)
3.5	7	1.5204916 8915603 6554754	(0)	-5.3136153 2415477 7787333	(-1)
4	8	1.5526081 5153521 3074366	(0)	-5.2327245 4238079 6631317	(-1)
4.5	9	1.5817760 4543288 2445714	(0)	-5.1596500 8142906 4228640	(-1)
5	10	1.6085245 5559983 5508655	(0)	-5.0931894 2457823 2664748	(-1)
6	12	1.6562600 7807588 5119687	(0)	-4.9763533 8641548 5935176	(-1)
7	14	1.6980407 3797943 6539694	(0)	-4.8763330 7835965 0759351	(-1)
8	16	1.7352810 4893716 5334385	(0)	-4.7891656 1495292 9018228	(-1)
9	18	1.7689373 6316879 8799245	(0)	-4.7120968 0940693 4039186	(-1)
10	1	3.0881849 3367358 4506057	(9)	-1.2705801 8012319 4551309	(10)
10	5	1.6763756 6094599 6762304	(5)	-2.7937076 6553618 0307492	(5)
10	10	3.0787321 6610908 3798699	(2)	-2.9192772 3808268 2287068	(2)
10	12	5.6013146 5285647 8179025	(1)	-4.2550940 4921594 7716229	(1)
10	15	8.5543948 7476833 4922279	(0)	-4.2060987 1253322 9228400	(0)
10	18	2.6897604 7548786 4071576	(0)	-7.3093180 6912417 6453080	(-1)
10	19	2.2814100 3147120 7820240	(0)	-5.1908367 8613721 4751736	(-1)
10	20	1.7996875 6697511 1865496	(0)	-4.6431493 8552189 3431386	(-1)
11	22	1.8280303 4491343 0501341	(0)	-4.5808619 4401178 4752499	(-1)
15	30	1.9237558 4878908 0421642	(0)	-4.3795124 4059142 7860014	(-1)
16	30	2.9152237 3020707 0519662	(0)	-6.0118800 5099136 2752760	(-1)
17	30	4.6888533 2510997 8143000	(0)	-1.3243102 2639360 1983398	(0)
18	30	8.8994996 0035960 3213215	(0)	-3.3920734 5465429 7680291	(0)
20	10	3.5866777 1567808 0657959	(10)	-6.0938358 0162628 6220833	(10)
20	20	9.1722767 8321025 5139368	(4)	-8.9396063 2184998 2957122	(4)
20	30	5.0586362 8619328 8651080	(1)	-2.7291466 5092312 0738696	(1)
22	30	4.3794340 3017728 9708807	(2)	-2.8701683 9774599 3230282	(2)
24	30	5.0956017 2444009 9636251	(3)	-3.8299843 3641932 4486710	(3)
25	30	1.9009942 7817409 0345328	(4)	-1.5115042 0942168 1596628	(4)
26	30	7.4694051 8982281 6077594	(4)	-6.2461887 6077957 2782673	(4)
28	30	1.3235672 5667906 5335018	(6)	-1.2080889 3050642 7091522	(6)
30	30	2.7533420 3795843 5267831	(7)	-2.7070286 7625464 4997837	(7)
32	60	4.2698844 2649648 5304731	(0)	-8.3772857 9320617 6026201	(-1)
32	64	2.1809034 2169040 9820925	(0)	-3.9047314 1376373 8830555	(-1)
32	68	4.0029619 8407414 5305352	(-1)	-5.1629512 4204759 9694083	(-1)