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The Asymptotic Expansion of a Hypergeometric Function ${}_2F_2(1, \alpha; \rho_1, \rho_2; z)$

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Abstract. The asymptotic expansion of a hypergeometric function ${}_2F_2(1, \alpha; \rho_1, \rho_2; z)$ is given in terms of hypergeometric functions ${}_2F_0(z^{-1})$ and ${}_3F_1(z^{-1})$.

Some years ago, the author [1] calculated the asymptotic expansion of a hypergeometric function ${}_2F_2(1, 1; 7/4, 9/4; z)$ in connection with a theory of thermolecular reaction kinetics. Recently, the author generalized it and obtained a simple asymptotic expansion of the function $F(z) = {}_2F_2(1, \alpha; \rho_1, \rho_2; z)$ with three independent parameters α, ρ_1 and ρ_2. The result may be written as follows:

$$ {}_2F_2(1, \alpha; \rho_1, \rho_2; z) \sim \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\alpha)} [K_{22}(z) + L_{22}(-z)], \qquad -\frac{3}{2}\pi < \arg z < \frac{\pi}{2}, $$

where α is neither a negative integer nor zero and

$$ K_{22}(z) = z^v e^z \, {}_2F_0(\rho_1 - \alpha, \rho_2 - \alpha; z^{-1}), \qquad v = 1 + \alpha - \rho_1 - \rho_2, $$

$$ L_{22}(z) = z^{-1} \frac{\Gamma(\alpha - 1)}{\Gamma(\rho_1 - 1)\Gamma(\rho_2 - 1)} \, {}_3F_1(1, 2 - \rho_1, 2 - \rho_2; 2 - \alpha; z^{-1}) $$

$$ + z^{-\alpha} \frac{\Gamma(\alpha)\Gamma(1 - \alpha)}{\Gamma(\rho_1 - \alpha)\Gamma(\rho_2 - \alpha)} \, {}_2F_0(1 + \alpha - \rho_1, 1 + \alpha - \rho_2; z^{-1}). $$

The general expression of $L_{22}(z)$ for the hypergeometric function ${}_2F_2(\alpha, \alpha'; \rho_1, \rho_2; z)$ with four parameters is well known [2], [3]. However, the corresponding $K_{22}(z)$ function is not explicitly known in general since it requires the solution of a three term recursion formula [2], [3]. For the proof of the present special result, it is sufficient to point out that the three term recursion formula given in [2] and [3] is satisfied by $(\rho_1 - \alpha)_k(\rho_2 - \alpha)_k/k!$ when account is taken of an obvious change of notation.*

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1. Shoon K. Kim, *J. Chem. Phys.*, v.46, 1967, p. 123.
2. Yudell L. Luke *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962, p. 14, Eq. (38). MR **25** #5198.
3. Yudell L. Luke, *The Special Functions and Their Approximations*. Vol. I, Math. in Sci. and Engineering, vol. 53, Academic Press, New York, 1969. MR **39** #3039.

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* An alternative proof was suggested by Yudell L. Luke in a private communication. From his recent work [3, p. 138, Eq. (12)], the function $F(z)$ satisfies $[(\delta + \rho_1 - 1)(\delta + \rho_2 - 1) - z(\delta + \alpha)]F(z) = (\rho_1 - 1)(\rho_2 - 1)$ and it is readily verified that $K_{22}(z)$ satisfies the homogeneous part of this equation.