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Diophantine Approximation of Ternary Linear Forms. II*

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Abstract. Let θ denote the positive root of the equation $x^3 + x^2 - 2x - 1 = 0$; that is, $\theta = 2\cos(2\pi/7)$. The main result of the paper is the evaluation of the constant $\limsup_{M\to\infty} \min M^2|x + \theta y + \theta^2 z|$, where the min is taken over all integers x, y, z satisfying $1 \leqq \max(|y|, |z|) \leqq M$. Its value is $(2\theta + 3)/7 \approx .78485$. The same method can be applied to other constants of the same type.

1. Introduction. Let θ denote the positive root of the equation $x^3 + x^2 - 2x - 1 = 0$; that is, $\theta = 2\cos(2\pi/7)$. The main result of this paper is the evaluation of the constant $\limsup_{M\to\infty} \min M^2\ |x + \theta y + \theta^2 z|$, where the min is taken over all integers x, y, z satisfying $1 \leqq \max(|y|, |z|) \leqq M$. Before going further, I shall indicate how this constant fits into the general theory of Diophantine approximations.

Dirichlet's well-known theorem on Diophantine approximation states that for any real number α and any positive integer M, there exist integers x and y satisfying

$$|\alpha y - x| < M^{-1}, \qquad 1 \leqq y \leqq M.$$

There are two n-dimensional generalizations of this result. First, for any real numbers α_i $(1 \leqq i \leqq n)$ and any positive integer M, there exist integers x_i $(1 \leqq i \leqq n+1)$ satisfying

$$\max_{1\leqq i\leqq n} |\alpha_i x_{n+1} - x_i| < M^{-1/n}, \qquad 1 \leqq x_{n+1} \leqq M. \tag{1}$$

Second, for any real numbers α_i $(1 \leqq i \leqq n)$ and any positive integer M, there exist integers x_i $(1 \leqq i \leqq n+1)$ satisfying

$$|x_{n+1} + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n| < M^{-n}, \qquad 1 \leqq \max_{1\leqq i\leqq n} |x_i| \leqq M. \tag{2}$$

Both of these theorems are immediate consequences of Minkowski's linear forms theorem.

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It is natural to ask whether, for given $\alpha_1, \cdots, \alpha_n$, either of these theorems can be improved. Only large values of M are of real interest, so we say that (1) is *improved by the constant* c ($c < 1$) for a given n-tuple $\alpha_1, \cdots, \alpha_n$ if for all sufficiently large M (1) holds with $M^{-1/n}$ replaced by $cM^{-1/n}$. Similarly, we say that (2) is *improved by the constant* c for a given n-tuple $\alpha_1, \cdots, \alpha_n$ if for all sufficiently large M (2) holds with M^n replaced by cM^n.

Now for any $\alpha_1, \cdots, \alpha_n$ define

$$c_1(\alpha_1, \cdots, \alpha_n) = \limsup_{M\to\infty} \min M^{1/n} \max_{1\leq i\leq n} |\alpha_i x_{n+1} - x_i|,$$

where the min is taken over all integers x_i ($1 \leqq i \leqq n+1$) satisfying $1 \leqq x_{n+1} \leqq M$, and

$$c_2(\alpha_1, \cdots, \alpha_n) = \limsup_{M\to\infty} \min M^n | x_{n+1} + \alpha_1 x_1 + \cdots + \alpha_n x_n|,$$

where the min is taken over all integers x_i ($1 \leqq i \leqq n+1$) satisfying

$$1 \leqq \max(|x_1|, \cdots, |x_n|) \leqq M.$$

It is clear that, for any $c < c_1(\alpha_1, \cdots, \alpha_n)$, (1) can be improved by the constant c, and that this is false for any $c > c_1(\alpha_1, \cdots, \alpha_n)$. Similarly, for any $c < c_2(\alpha_1, \cdots, \alpha_n)$, (2) can be improved by the constant c, and this is false for any $c > c_2(\alpha_1, \cdots, \alpha_n)$.

In the case $n = 1$, $c_1(\alpha) = c_2(\alpha)$ and it is possible, by the use of continued fractions, to completely solve the problem of evaluating $c_1(\alpha)$.

Let $[a_0, a_1, a_2, \cdots]$ denote the simple continued fraction expansion of the real number α. Then, independently and at about the same time, Davenport and Schmidt [3, Theorem 1, pp. 113–114] and Lesca [6, p. 61] proved that

$$(3) \qquad c_1(\alpha) = \left(1 + \liminf_{n\to\infty} [0, a_{n+1}, a_{n+2}, \cdots]\cdot[0, a_n, a_{n-1}, \cdots, a_1]\right)^{-1}.$$

Lesca [6, Chapter 3, pp. 57–72] carried the study of $c_1(\alpha)$ further. He showed that there is an infinite sequence of values of $c_1(\alpha)$ between its smallest possible value $(5+\sqrt{5})/10$ (attained for $\alpha = \frac{1}{2}(1+\sqrt{5})$) and $(1+\sqrt{5})/4$. This sequence of successive minimal values of $c_1(\alpha)$ is analogous to the more familiar sequence of successive minimal values of $\liminf_{y\to\infty} |y(\alpha y - \{\alpha y\})|$ (here $\{\alpha y\}$ denotes the nearest integer to αy) first described by Markoff (see, for example, Chapter II of the book by Cassels [1]). Lesca proved various further results about the sequence of minimal values of $c_1(\alpha)$, analogous to some of Markoff's theorems.

It follows immediately from the formula (3) that $c_1(\alpha) = 1$ for almost all α, and that $c_1(\alpha) < 1$ if and only if the partial quotients a_i in the continued fraction for α are bounded, i.e., α is "badly approximable". This suggests an interesting question: For $n > 1$, what is the largest constant c_n such that $c_1(\alpha_1, \cdots, \alpha_n) \geqq c_n$ holds for almost all n-tuples $\alpha_1, \cdots, \alpha_n$? This problem was posed by Jarnik [5] in 1958; I am not aware of any earlier references to the problem.

There was no progress on this problem until Davenport and Schmidt [4] proved that, for each $n \geqq 1$, $c_1(\alpha_1, \cdots, \alpha_n) = 1$ for almost all $\alpha_1, \cdots, \alpha_n$. They also showed that $c_2(\alpha_1, \cdots, \alpha_n) = 1$ for almost all $\alpha_1, \cdots, \alpha_n$.

Davenport and Schmidt [3, Theorem 2, p. 115] also proved that, for each $n > 1$, a sufficient (but not necessary) condition for the constants $c_1(\alpha_1, \cdots, \alpha_n)$ and

$c_2(\alpha_1, \cdots, \alpha_n)$ to be less than 1 is that $\alpha_1, \cdots, \alpha_n$ is a badly approximable n-tuple. (An n-tuple $\alpha_1, \cdots, \alpha_n$ is said to be badly approximable if there is some constant $c > 0$ such that

$$\max_{1 \leqq i \leqq n} |\alpha_i x_{n+1} - x_i| > c\, |x_{n+1}|^{-1/n}$$

for all integers x_i $(1 \leqq i \leqq n + 1)$ with $x_{n+1} \neq 0$; or, equivalently, if there is some constant $c > 0$ such that

$$|x_{n+1} + \alpha_1 x_1 + \cdots + \alpha_n x_n| > c\Big(\max_{1 \leqq i \leqq n} |x_i|\Big)^{-n}$$

for all integers x_i $(1 \leqq i \leqq n + 1)$ with $x_1, \cdots, x_n$ not all zero.) Since any numbers $\alpha_1, \cdots, \alpha_n$ in a real algebraic number field of degree $n + 1$ such that $1, \alpha_1, \cdots, \alpha_n$ are linearly independent over the rationals make up a badly approximable n-tuple (see Cassels [1, pp. 79–80]), it follows that the constant $c_2(\theta, \theta^2)$, with which this paper is concerned, is less than 1. Davenport and Schmidt [3, pp. 122–126] proved that $c_2(\theta, \theta^2) < 10/11 = .90909 \ldots$.

The obstacle in the way of an exact evaluation of $c_2(\theta, \theta^2)$ is, of course, the absence of a continued fraction algorithm, which was essential in deriving (3). However, it turns out that the algorithm introduced in my paper [2] for the purpose of finding integer solutions x, y, z of the inequality

$$|x + \alpha y + \beta z| \max(y^2, z^2) < c,$$

where α and β are algebraic integers in a totally real cubic field and c is a small constant, has many features similar to those of the simple continued fraction algorithm. In fact, the algorithm of [2] makes it possible to evaluate $c_2(\theta, \theta^2)$ via some moderately lengthy computations.

2. Some Preliminaries. I begin by giving a brief exposition of the application of the method of my paper [2] to the inequality

$$|x + \theta y + \theta^2 z| \max(y^2, z^2) < 1.3. \tag{4}$$

For a detailed account of the method and proofs of various assertions made here, the reader should refer to [2].

Let F denote the cubic field defined by θ, and let $\theta' = 2\cos(4\pi/7)$, $\theta'' = 2\cos(6\pi/7)$ be the conjugates of θ. Then (note that all decimals in this paper are truncated, not rounded off)

$$\theta = 1.24697960 \ldots, \quad \theta' = -.44504186 \ldots, \quad \theta'' = -1.80193773 \ldots .$$

Since F is a cyclic or Abelian field, θ' and θ'' belong to F. Also, $1, \theta, \theta^2$ is an integral basis for F.

The field F and the linear form $x + \theta y + \theta^2 z$ were used as an example in [2, Section 6], so I simply state the results obtained there.

Let $\varphi = 1/\theta'$ as in [2, Section 6], so θ, φ is a pair of fundamental units for F. If ω is any unit of norm 1 in F, let $Q(\omega)$ denote the 3 by 3 integer matrix which satisfies

$$[1\ \theta\ \theta^2]Q(\omega) = [\omega\ \omega\theta\ \omega\theta^2]$$

(in the terminology of [2, p. 166], $Q(\omega)$ takes $x + \theta y + \theta^2 z$ to $\omega(x + \theta y + \theta^2 z)$). Thus

$$Q(\theta) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \qquad Q(\varphi) = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}.$$

The set of matrices $\{Q(\omega)$: ω is a unit with norm 1 in $F\}$ is a commutative group under matrix multiplication with generators $Q(\theta)$ and $Q(\varphi)$ [2, Lemma 1, p. 167].

Given any unit ω of norm 1 in F, $\omega^{-1} = \theta^m \varphi^n$ for some unique integers m and n. Define $R(m, n) = \theta\omega^{-1}$; then [2, formula (13), p. 169],

$$|R(m, n)| = |b_m^{(n)} + g_m^{(n)}\theta + k_m^{(n)}\theta^2|,$$

where $b_m^{(n)}$, $g_m^{(n)}$, $k_m^{(n)}$ are the entries of the middle column of $Q(\theta^m\varphi^n)$, read from top to bottom. Further define

$$S(m, n) = |R(m, n)| \max((g_m^{(n)})^2, (k_m^{(n)})^2). \tag{5}$$

For each integer n, let $v(n)$ denote the value of m with the property that $S(v(n), n) < S(m, n)$ for all integers $m \neq v(n)$. If, as in [2, Section 4], the values of $S(m, n)$ are tabulated in a rectangular array with the integers m arranged on a vertical axis and the integers n arranged on a horizontal axis, then $S(v(n), n)$ is the smallest entry in one of the columns of the array. The second quadrant ($m \geqq 0$, $n < 0$) of the array is the only portion which is of interest for the linear form $x + \theta y + \theta^2 z$ [2, formula (19), p. 171]. For the convenience of the reader, Table 1 of [2], which gives part of the second quadrant of the array for $n \geqq -40$, is reproduced in this paper.

It is proved in [2, Section 7] that if $(x, y, z) = (b, g, k)$ is a solution of $|x + \theta y + \theta^2 z| \max(y^2, z^2) < .187N$, where N is the smallest value larger than 1 which can be taken by the norm of $x + \theta y + \theta^2 z$, then, except possibly for a finite number of exceptions, $(b, g, k) = (b_m^{(n)}, g_m^{(n)}, k_m^{(n)})$ for some integers $m \geqq 0, n < 0$. It is easy to see that $N \geqq 7$, for a computation gives

$$\begin{aligned} \text{Norm } (x + \theta y + \theta^2 z) = {} & x^3 + y^3 + z^3 - x^2 y + 5x^2 z \\ & - 2xy^2 + 6xz^2 - y^2 z - 2yz^2 - xyz \end{aligned}$$

and simple congruence considerations show that the right-hand side is divisible by 2, 3 or 5 if and only if each of x, y, z is divisible by 2, 3 or 5, respectively. Hence, except possibly for a finite number of exceptions, every solution of (4) corresponds to some $S(m, n)$ with $m \geqq 0$, $n < 0$.

If α is any function of θ, θ' and θ'', let α' and α'' denote the numbers obtained by replacing $\theta, \theta', \theta''$ by $\theta', \theta'', \theta$ and $\theta'', \theta, \theta'$, respectively, in the expression for α. Thus if $\omega^{-1} = \theta^m\varphi^n$, then $\omega'^{-1} = \theta'^m\varphi'^n$ and $\omega''^{-1} = \theta''^m\varphi''^n$. Now let n be any fixed integer and put $\omega_m^{-1} = \theta^m\varphi^n$; define $u(n)$ to be the value of m with the property that

$$\big|\, |\omega''_{u(n)}/\omega'_{u(n)}| - 1 \big| < \big|\, |\omega''_m/\omega'_m| - 1 \big|$$

for all integers $m \neq u(n)$. The function $u(n)$ is easy to calculate, as the following lemma [2, Lemma 5, p. 170] shows:

LEMMA 1. *Define* $E_1 = \theta\theta'^2$ *and* $E_2 = |\varphi\varphi'^2|$. *The integer* $u(n)$ *is equal to the unique integer* m *which satisfies*

$$\frac{\log(2(1 + E_1)^{-1})}{\log E_1} < m + \frac{n \log E_2}{\log E_1} < 1 + \frac{\log(2(1 + E_1)^{-1})}{\log E_1}. \tag{6}$$

TABLE 1. *Second Quadrant of S(m, n) Array*

2.47	4.08	5.38	8.29	11.5	17.1	24.2	35.4	50.7	73.8	106	154	222	321	463	670	969	1393	2042		7
.524	1.18	1.18	2.21	2.68	4.38	5.81	8.91	12.4	18.4	26.0	38.1	54.6	79.4	114	166	238	347	491		6
.220*	.493*	.323*	.726*	.556	1.25	1.28	2.36	2.90	4.70	6.27	9.58	13.3	19.8	28.1	41.0	59.0	84.8	126		5
.204	.459	.188	.423	.226*	.508*	.339*	.762*	.591	1.33	1.40	2.53	3.14	5.05	6.77	10.3	14.3	21.5	29.3		4
.912	.995	.343	.609	.200	.450	.190	.426	.233*	.525*	.357*	.801*	.629	1.41	1.52	2.70	3.41	5.33	7.66	9.68	3
4.30	3.33	1.95	1.71	.833	.941	.307	.585	.197	.442	.191	.430	.242*	.543*	.377*	.846	.684	1.54	1.54	3.45	2
18.0	12.8	8.52	6.26	3.97	3.11	1.80	1.60	.760	.891	.273	.563	.194	.435	.193	.434*	.244*	.548*	.308*	.692	1
73.6	51.3	35.1	24.7	16.7	11.9	7.90	5.82	3.68	2.90	1.65	1.51	.693	.845	.242	.544	.196	.440	.247	.555*	0
−20	−19	−18	−17	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	n \ m
.395*	.887	.754	1.60	1.78	3.09	3.95	6.22	8.47	12.7	17.9	26.4	37.6	54.9	78.8	114	165	239	344	498	11
.196	.441*	.260*	.583*	.417*	.937	.827	1.70	1.94	3.31	4.27	6.68	9.14	13.7	19.3	28.4	40.5	59.1	84.9	123	10
.234	.526	.190	.426	.200	.449*	.270*	.607*	.441*	.990	.905	1.81	2.10	3.55	4.61	7.18	9.86	14.7	20.8	30.6	9
1.41	1.33	.575	.765	.227	.510	.188	.423	.204	.458*	.282*	.633*	.466*	1.05	.989	1.94	2.28	3.80	4.98	7.71	8
6.32	4.73	2.92	2.38	1.29	1.26	.521	.729	.220	.495	.188	.422	.208	.468*	.294*	.661*	.494	1.11	1.08	2.07	7
26.2	18.5	12.4	8.97	5.85	4.41	2.69	2.22	1.19	1.18	.471	.695	.214	.481	.187	.421	.214*	.480*	.308*	.692*	6
107	74.2	51.0	35.7	24.3	17.2	11.5	8.34	5.42	4.11	2.49	2.08	1.09	1.11	.425	.664	.209	.469	.188	.422	5
433	300	207	143	99.1	68.9	47.3	33.1	22.6	16.0	10.7	7.76	5.02	3.83	2.29	1.95	.997	1.05	.382	.635	4
−40	−39	−38	−37	−36	−35	−34	−33	−32	−31	−30	−29	−28	−27	−26	−25	−24	−23	−22	−21	n \ m

The numbers S(u(n), n) are indicated by asterisks.

Let $E(n)$ denote $\omega''_{u(n)}/\omega'_{u(n)} = (\theta\theta'^2)^{u(n)}(\varphi\varphi'^2)^n$; this function will be very important later on. Note that it follows simply from the definition of $u(n)$ that [2, formula (17), p. 170]

$$(7) \qquad 2\theta^{-1} > |E(n)| > 2\theta'^2$$

for every n. So we define $E_+ = 2\theta^{-1}$ and $E_- = 2\theta'^2$.

By [2, Theorem 1, p. 171], for every integer $n < 0$,

$$r_1 \geqq v(n) - u(n) \geqq r_2,$$

where r_1 and r_2 are integers which can be calculated.

The calculation of r_1 and r_2 depends on the following considerations. Define $T_{m,n} = \max(|g^{(n)}_{u(n)+m}|, |k^{(n)}_{u(n)+m}|)$. It is easily seen [2, remarks preceding Lemma 6] that $S(u(n) + m, n) < S(u(n) + m + 1, n)$ holds if and only if

$$(8) \qquad T_{m+1,n}/T_{m,n} > \theta^{-1/2};$$

and that, for each n, $T_{m+1,n}/T_{m,n} \to |\theta''| > \theta^{-1/2}$ as $m \to +\infty$ and $T_{m+1,n}/T_{m,n} \to |\theta'| < \theta^{-1/2}$ as $m \to -\infty$. Thus, for a given n, (8) is true for all sufficiently large m and false for all sufficiently large $|m|$, $m < 0$. If integers m_+ and m_- can be found with the property that, for each $n < 0$, (8) holds for all $m \geqq m_+$ and (8) is false for all $m \leqq m_- - 1$, then we may clearly take $r_1 = m_+$ and $r_2 = m_-$.

A method for finding m_+ and m_- is given in [2, Lemma 6]. The following notations simplify the explanation of this procedure: For any m and n, $T_{m+1,n}/T_{m,n}$ is equal to one of the four quotients

$$|g^{(n)}_{u(n)+m+1}/g^{(n)}_{u(n)+m}|, \quad |g^{(n)}_{u(n)+m+1}/k^{(n)}_{u(n)+m}|, \quad |k^{(n)}_{u(n)+m+1}/k^{(n)}_{u(n)+m}|, \quad |k^{(n)}_{u(n)+m+1}/g^{(n)}_{u(n)+m}|;$$

we say $T_{m+1,n}/T_{m,n}$ is of *type* 1, 2, 3 or 4, respectively. Next we define

$$(9) \qquad G = \tfrac{1}{7}\theta(\theta'^2 - \theta''^2), \qquad K = \tfrac{1}{7}\theta(\theta'' - \theta'),$$

and further define two numbers $I = I(m, n)$ and $J = J(m, n)$ in F, each of which depends on the type of $T_{m+1,n}/T_{m,n}$, as follows: If the type of $T_{m+1,n}/T_{m,n}$ is 1, 2, 3 or 4, then (I, J) equals (G, G), (G, K), (K, K) or (K, G), respectively.

Define the function $f_t(m, E)$ for $t = 1, 2, 3, 4$ by

$$f_t(m, E) = |\theta' E I' \theta'^m + \theta'' I'' \theta''^m| / |E J' \theta'^m + J'' \theta''^m|$$

where I and J have the values which they take on for type t and E is a parameter which satisfies $E_+ > |E| > E_-$.

It is proved in [2, Lemma 6] that for each sufficiently large fixed $|n|$, $n < 0$, the inequality (8) is false for some m only if $f_t(m, E) < \theta^{-1/2}$ is possible for some choice of t and E. A calculation shows that $|\theta'' I''/J''| > \theta^{-1/2}$ for each t, so $f_t(m, E)$ must exceed $\theta^{-1/2}$ for large enough m. This implies (8) is true for large enough m, and so gives an upper bound on $v(n) - u(n)$. Carrying out the calculations for each t gives the following:

$$(10) \qquad \begin{aligned} &f_t(0, E) < \theta^{-1/2} && \text{is not possible for } t = 1, 3, 4, \\ &f_2(0, E) < \theta^{-1/2} && \text{is possible, but only for } E < 0, \\ &f_2(1, E) < \theta^{-1/2} && \text{is not possible.} \end{aligned}$$

It follows that $1 \geqq v(n) - u(n)$ for $|n|$ large enough, $n < 0$.

It is also proved in [2, Lemma 6] that, for each sufficiently large fixed $|n|$, $n < 0$, the inequality (8) is true for some m only if $f_t(m, E) > \theta^{-1/2}$ is possible for some choice of t and E. Since $|\theta' I'/J'| < \theta^{-1/2}$ for each t, $f_t(m, E)$ is less than $\theta^{-1/2}$ for sufficiently large $|m|$, $m < 0$. Thus we can calculate a lower bound for $v(n) - u(n)$ as follows:

$$\begin{aligned} &f_t(-1, E) > \theta^{-1/2} \quad \text{is possible for } t = 1, 2, 3, 4,\\ (11)\qquad &f_t(-2, E) > \theta^{-1/2} \quad \text{is possible only for } t = 1, 2, 4,\\ &f_t(-3, E) > \theta^{-1/2} \quad \text{is possible only for } t = 4. \end{aligned}$$

It follows that $v(n) - u(n) \geqq -3$ for $|n|$ large enough, $n < 0$.

In fact, $1 \geqq v(n) - u(n) \geqq -3$ holds for all $n < 0$, as was remarked in [2, p. 177]. For the purposes of this paper, it is convenient to improve on these inequalities.

THEOREM 1. *For each even integer $n < 0$, $u(n)$ equals either $v(n) + 1$ or $v(n) + 2$. For each odd integer $n < 0$, $u(n)$ equals either $v(n)$ or $v(n) + 1$.*

Proof. Define the function $g(m, E)$ by

$$g(m, E) = \left| \frac{EG'\theta'^m + G''\theta''^m}{EK'\theta'^m + K''\theta''^m} \right| ,$$

where G and K are the constants defined by (9) and E is a parameter which satisfies $E_+ > |E| > E_-$.

By an argument similar to that used in [2, Lemma 6], we find that for each sufficiently large fixed $|n|$, $n < 0$, the inequality

$$|g^{(n)}_{u(n)+m}/k^{(n)}_{u(n)+m}| < 1$$

is false for some m only if $g(m, E) > 1$ is possible for some choice of E. A calculation shows that $g(m, E)$ tends to a constant less than 1 as $m \to +\infty$ and as $m \to -\infty$, and a little more computation gives

$$\begin{aligned} g(i, E) &> 1 \quad \text{holds for } i = 0, -1 \text{ if and only if } E > 0,\\ g(2, E) &> 1 \quad \text{is not possible},\\ g(+1, E) &> 1 \quad \text{is possible, but only for } E > |\theta^3\theta'|^{-1} = 1.1588 \ldots ,\\ g(-2, E) &> 1 \quad \text{is possible, but only for } E \text{ satisfying } |\theta''|^{-1} = .5549 \ldots > E > 0,\\ g(-3, E) &> 1 \quad \text{is not possible}. \end{aligned}$$

It follows immediately that, for all sufficiently large $|n|$, $n < 0$,

$$\begin{aligned} &T_{m,n} = |k^{(n)}_{u(n)+m}| \quad \text{for each } m \leqq -3 \text{ and each } m \geqq 2,\\ &T_{1,n} = |g^{(n)}_{u(n)+1}| \quad \text{is only possible if } E > |\theta^3\theta'|^{-1},\\ (12)\qquad &T_{i,n} = |g^{(n)}_{u(n)+i}|, \quad E(n) > 0, \qquad (i = 0, -1),\\ &T_{i,n} = |k^{(n)}_{u(n)+i}|, \quad E(n) < 0,\\ &T_{-2,n} = |g^{(n)}_{u(n)-2}| \quad \text{is only possible if } |\theta''|^{-1} > E > 0. \end{aligned}$$

Referring to (10), we see that $1 = v(n) - u(n)$ could occur only if $T_{1,n} = |g^{(n)}_{u(n)+1}|$, $T_{0,n} = |k^{(n)}_{u(n)}|$ and $E < 0$; however, this would contradict the second statement in

(12), so we may conclude that $0 \geqq v(n) - u(n)$ for all sufficiently large $|n|$, $n < 0$.

Similarly, (11) implies that $v(n) - u(n) = -3$ could occur only if $T_{-2,n} = |k_{u(n)-2}^{(n)}|$ and $T_{-3,n} = |g_{u(n)-3}^{(n)}|$, which would contradict the first statement in (12). Therefore $v(n) - u(n) \geqq -2$ for all sufficiently large $|n|$, $n < 0$.

In order to establish that $u(n) = v(n)$ does not hold for some integer n, it suffices to show that (8) holds with $m = -1$ for the given n. It is proved in [2, Lemma 6] that, for each sufficiently large fixed $|n|$, $n < 0$, the inequality (8) is true with $m = -1$ if $f_t(-1, E(n)) > \theta^{-1/2}$ is true, where t is the type of $T_{0,n}/T_{-1,n}$. Since $E(n)$ is positive if and only if n is even, it follows from the third statement in (12) that $u(n) = v(n)$ is false for each sufficiently large even $|n|$, $n < 0$, provided $f_1(-1, E) > \theta^{-1/2}$ is true for any allowable positive value of E (which can be verified by a simple calculation).

If $u(n) = v(n) + 2$ for some n, then (8) is true with $m = -2$; it then follows from the second statement in (11) that, if $|n|$ is sufficiently large and $n < 0$, $T_{-1,n}/T_{-2,n}$ is of type 1, 2 or 4. Since $E(n)$ is positive if and only if n is even, the third statement in (12) implies that $T_{-1,n}/T_{-2,n}$ can be of type 1 or 2 only if n is even, and the third and fourth statements in (12) imply that $T_{-1,n}/T_{-2,n}$ cannot be of type 4. Thus for $|n|$ sufficiently large, $n < 0$, we can have $u(n) = v(n) + 2$ only for n even.

This completes the proof of the theorem for sufficiently large $|n|$, and it is easily verified that in fact the theorem holds for every $n < 0$ (see [2, Tables 1 and 2]).

The following simple lemma summarizes some useful facts about the function $u(n)$.

LEMMA 2. *Suppose $n < 0$; then $u(n)$ increases as n decreases. There are no more than four consecutive values of n for which $u(n)$ has the same value. Whenever $u(n)$ takes on a given value, it has that value for at least three consecutive values of n. For $n < 0$, $u(n)$ takes on every integer value $\geqq 0$.*

Proof. A little calculation using (6) in Lemma 1 gives

$$u(n-4) \geqq u(n) + 1 \geqq u(n-1) \geqq u(n)$$

for every $n < 0$. We also find that

$$u(n-1) = u(n) + 1 \quad \text{implies } u(n-1) = u(n-2) = u(n-3).$$

Putting these facts together gives the lemma.

The pattern of values of $u(n)$, $-1 \geqq n \geqq -40$, is easily seen in Table 1, where the numbers $S(u(n), n)$ are indicated by asterisks.

3. Evaluation of $c_2(\theta, \theta^2)$. Since θ, θ^2 is a badly approximable pair (see the Introduction), there is a positive lower bound for the numbers $S(m, n)$. The following result [2, Lemma 10, p. 177] gives the value of the smallest constant c such that, given any $\epsilon > 0$, the inequality $S(m, n) < c + \epsilon$ holds infinitely often. This constant is plainly equal to $\liminf_{n\to-\infty} S(v(n), n)$.

LEMMA 3. *Let $\theta = 2\cos(2\pi/7)$; then*

$$\liminf_{M\to\infty} \min_{M} |x + \theta y + \theta^2 z| \max(y^2, z^2) = \tfrac{4}{49}(\theta^2 + 3\theta - 3) = .1874\ldots,$$

where the min *is taken over all integers x, y, z such that* $\max(|y|, |z|) = M$.

The following lemma indicates, among other things, that $S(u(n) - 1, n)$ is arbitrarily near the constant of Lemma 3 for suitable *even* values of n.

LEMMA 4. *Let $E(n) = (\theta\theta'^2)^{u(n)}(\varphi\varphi'^2)^n$. If $n \to -\infty$ through even values of n, then*

(13) $$S(u(n)+i,n)-\theta^{i+1}(\theta'^{i}G'\,|E(n)|^{1/2}+\theta''^{i}G''\,|E(n)|^{-1/2})^2\to 0$$

$(i = 0$ *or* $-1)$.

If $n\to-\infty$ *through odd values of* n, *then*

(14) $$S(u(n)+i,n)-\theta^{i+1}(\theta'^{i}K'\,|E(n)|^{1/2}-\theta''^{i}K''\,|E(n)|^{-1/2})^2\to 0$$

$(i = 0$ *or* $-1)$.

Proof. Define $\omega_{u(n)}^{-1} = \theta^{u(n)}\varphi^n$. It follows from the definition of $R(m, n)$ that, for any integer i, $R(u(n)+i, n) = \theta^{i+1}\omega_{u(n)}^{-1}$, so (5) implies

(15) $$S(u(n)+i,n)=\theta^{i+1}\,|\omega_{u(n)}^{-1}|\,\max((g_{u(n)+i}^{(n)})^2,(k_{u(n)+i}^{(n)})^2)$$

for any integer i.

It is obvious that $E(n)$ is positive if and only if n is even; hence the third statement in (12), (15) and [2, formula (16) with $p = u(n)$, $m = 0$ or -1] imply that

(16) $$S(u(n)+i,n)=\frac{\theta^{i+1}}{|\omega_{u(n)}|}\left(\frac{\theta^{i}H}{\omega_{u(n)}}+\frac{\theta'^{i}H'}{\omega'_{u(n)}}+\frac{\theta''^{i}H''}{\omega''_{u(n)}}\right)^2 \qquad (i=0,-1),$$

where the constant H is equal to either G (if n is even) or K (if n is odd).

It is easily seen that $E(n) = \omega_{u(n)}\omega''^2_{u(n)} = (\omega_{u(n)}\omega'^2_{u(n)})^{-1}$ and $|\omega_{u(n)}|\to\infty$ as $n\to-\infty$. These facts in conjunction with (16) imply the lemma.

If $i = -1$, the functions appearing in (13) and (14) are important enough to be given names; define

$$\tau(E)=(E^{1/2}G'/\theta'+E^{-1/2}G''/\theta'')^2,\qquad \kappa(E)=(E^{1/2}K'/\theta'-E^{-1/2}K''/\theta'')^2.$$

The minimum value of $\tau(E)$ occurs for $E = \theta^{-1}$, and $\tau(\theta^{-1})$ is just the constant of Lemma 3 (see Figure 1). Hence

(17) $$\liminf_{n\to-\infty} S(v(n),n)=\liminf_{n\text{ even}} S(u(n)-1,n).$$

This relation does not, of course, mean that $v(n) = u(n) - 1$ for n even. It can be shown that the equality $v(n) = u(n) - 2$, which is allowable for n even by Theorem 1, does occur, although very rarely.

We now turn to our main result.

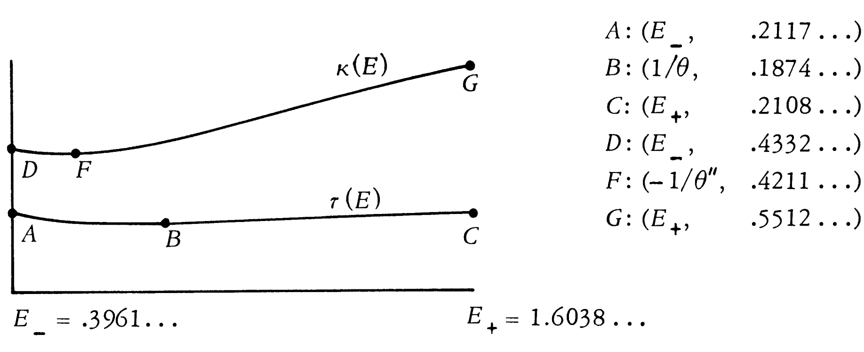

FIGURE 1

THEOREM 2. *Let* $\theta = 2\cos(2\pi/7)$; *then*

$$\limsup_{M\to\infty} \min M^2 \, |x + \theta y + \theta^2 z| = \tfrac{1}{7}(2\theta + 3) = .78485\ldots, \tag{18}$$

where the min *is taken over all integers* x, y, z *satisfying* $1 \leqq \max(|y|, |z|) \leqq M$.

Because of (17), the integers $T^2_{-1,n}$, n even, will play a special role in what follows. We first require a lemma giving bounds on the ratio $T^2_{-1,n-2}/T^2_{-1,n}$, n even.

LEMMA 5. *If* $n \to -\infty$ *through even values of* n *such that* $u(n-2) = u(n)$, *then*

$$\frac{T_{-1,n-2}}{T_{-1,n}} - 1 + \frac{E(n)G'/\theta'^2 + G''/\theta''^2}{E(n)G'/\theta' + G''/\theta''} \to 0 \tag{19}$$

and

$$\limsup \frac{T^2_{-1,n-2}}{T^2_{-1,n}} = 5.6773\ldots, \qquad \liminf \frac{T^2_{-1,n-2}}{T^2_{-1,n}} = 4.4707\ldots. \tag{20}$$

If $n \to -\infty$ *through even values of* n *such that* $u(n-2) = u(n) + 1$, *then*

$$\frac{T_{-1,n-2}}{T_{-1,n}} - 1 + \frac{E(n)G' + G''}{E(n)G'/\theta' + G''/\theta''} \to 0 \tag{21}$$

and

$$\limsup \frac{T^2_{-1,n-2}}{T^2_{-1,n}} = 4.5715\ldots, \qquad \liminf \frac{T^2_{-1,n-2}}{T^2_{-1,n}} = 3.5998\ldots. \tag{22}$$

Proof. Note that by Lemma 2 the only possible values for $u(n-2)$ are $u(n)$ and $u(n)+1$. By the third statement in (12), $T_{-1,n} = |g^{(n)}_{u(n)-1}|$. The same statement plus a little manipulation with the matrices in [2, Section 3] gives

$$T_{-1,n-2} = |b^{(n)}_{u(n)-1} - g^{(n)}_{u(n)-1} + k^{(n)}_{u(n)-1}| \quad \text{if } u(n-2) = u(n)$$

and

$$T_{-1,n-2} = |b^{(n)}_{u(n)-1} - g^{(n)}_{u(n)-1} + 2k^{(n)}_{u(n)-1}| \quad \text{if } u(n-2) = u(n) + 1.$$

Using [2, formula (16) with $p = u(n)$, $m = -1$], we can calculate functions of $E(n)$ to which $b^{(n)}_{u(n)-1}/g^{(n)}_{u(n)-1}$ and $k^{(n)}_{u(n)-1}/g^{(n)}_{u(n)-1}$ tend as $n \to -\infty$ through even values; we use the same facts about $E(n)$ and $\omega_{u(n)}$ that were applied to (16) in the proof of Lemma 4. A little more calculation gives (19) and (21).

To prove (20) and (22), we need only find the maximum and minimum of the functions of $E(n)$ in (19) and (21), subject to the constraints

$$\begin{aligned} &u(n-2) = u(n) &&\text{if and only if } E_- < |E(n)| < 2\theta^3\theta'^2 = .7680\ldots, \\ &u(n-2) = u(n) + 1 &&\text{if and only if } 2\theta^3\theta'^2 < |E(n)| < E_+. \end{aligned} \tag{23}$$

To prove (23), we use the fact that $E(n) = \theta^{u(n)+2n}\theta'^{2u(n)+n}$. Thus $E(n-2) = E(n)/\theta^4\theta'^2 \approx 2.0881E(n)$ if $u(n-2) = u(n)$ and $E(n-2) = E(n)/\theta^3 \approx .5157E(n)$ if $u(n-2) = u(n)+1$; hence (7) implies (23).

For future reference, define

$$h_1(E) = \frac{-(EG'/\theta'^2 + G''/\theta''^2)}{EG'/\theta' + G''/\theta''}, \qquad h_2(E) = \frac{-(EG' + G'')}{EG'/\theta' + G''/\theta''}.$$

Figure 2 gives graphs of these functions.

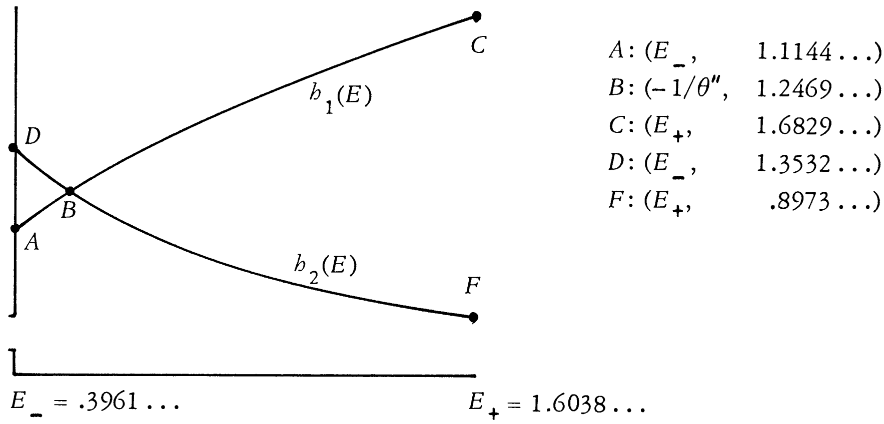

FIGURE 2

It may be appropriate to mention here that $|E(n)|$ $(n = -1, -2, \cdots)$ is dense in the interval $[E_-, E_+]$ (this follows easily from Kronecker's theorem on inhomogeneous Diophantine approximation [1, p. 53]). This fact will be used several times later on, without explicit mention (e.g., in the discussion after Lemma 6 in this section).

It follows from (13) that

$$\limsup_{n\to-\infty;\, n \text{ even}} T^2_{-1,n}\, |b^{(n)}_{u(n)-1} + g^{(n)}_{u(n)-1}\theta + k^{(n)}_{u(n)-1}\theta^2| = \max_{E_- \leqq E \leqq E_+} \tau(E) = .2117\ldots, \tag{24}$$

so we may assume in (18) that $T^2_{-1,n-2} > M^2 > T^2_{-1,n}$, n even, say $M^2 = \alpha T^2_{-1,n}$, $\alpha > 1$. Hence

$$T^2_{-1,n-2}/T^2_{-1,n} > \alpha > 1. \tag{25}$$

It is clear from (24) that the lim sup in (18) will be approached only for those M for which α is reasonably large. Lemma 5 gives upper bounds on the size of α, because of (25).

Since all solutions of (4) correspond to some $S(m, n)$ and we know $c_2(\theta, \theta^2) < 1$, in evaluating min $M^2\, |x + \theta y + \theta^2 z|$ in (18), where $M^2 = \alpha T^2_{-1,n}$ and (25) holds, it suffices to consider only those integer triples (x, y, z) which are equal to $(b^{(n+j)}_{u(n+j)+m}, g^{(n+j)}_{u(n+j)+m}, k^{(n+j)}_{u(n+j)+m})$ for some integers j and m. Given such a triple, we have

$$M^2\, |x + \theta y + \theta^2 z| = \alpha T^2_{-1,n}\, |R(u(n + j) + m, n + j)|$$

and

$$M^2 = \alpha T^2_{-1,n} \geqq T^2_{m,n+j}. \tag{26}$$

Combining (25) and (26) gives

$$T^2_{-1,n-2}/T^2_{-1,n} > \alpha \geqq T^2_{m,n+j}/T^2_{-1,n}. \tag{27}$$

Thus the problem of evaluating the constant in (18) is reduced to that of evaluating

$$\limsup_{n\to-\infty} \min_{j,m} \sup_{\alpha} \alpha T^2_{-1,n}\, |R(u(n + j) + m, n + j)|, \tag{28}$$

where j and m are any integers and α satisfies (27). The remainder of this section is devoted to evaluating (28).

By the definitions $R(m, n) = \theta\omega_m^{-1}$ and $\omega_m^{-1} = \theta^m\theta'^{-n}$,

$$|R(u(n+j)+m, n+j)| = \theta^{m+1+u(n+j)}\,|\theta'|^{-n-j}$$
$$= \theta^{u(n+j)-u(n)+m+1}\,|\theta'|^{-j}\,|R(u(n)-1, m)|.$$

Hence, by Lemma 4 with $i = -1$, if $n \to -\infty$ through even values of n, then

$$T_{-1,n}^2\,|R(u(n+j)+m, n+j)| - \theta^{u(n+j)-u(n)+m+1}\,|\theta'|^{-j}\tau(E(n)) \to 0; \tag{29}$$

if $n \to -\infty$ through odd values of n, then

$$T_{-1,n}^2\,|R(u(n+j)+m, n+j)| - \theta^{u(n+j)-u(n)+m+1}\,|\theta'|^{-j}\,\kappa(|E(n)|) \to 0.$$

Calculations based on (29) will give the following lemma.

LEMMA 6. *Suppose $n < 0$, n even. If $|n|$ is sufficiently large, then*

$$\min_{j,m}\sup_{\alpha}\alpha T_{-1,n}^2\,|R(u(n+j)+m, n+j)|, \tag{30}$$

where j and m are any integers and α satisfies (27), *is given by the following integer pairs j, m:*

(A) $j = 0,\ m = -1$ *if* $-3\theta^2 + 3\theta + 2 = 1.0760\ldots < E(n) < E_+$,

(B) $j = 0,\ m = -2$ *if* $2\theta^3\theta'^2 = .7680\ldots < E(n) < -3\theta^2 + 3\theta + 2$ *or* $E_- < E(n) < |\theta''|^{-1} = .5549\ldots$,

(C) $j = -2,\ m = 0$ *if* $|\theta''|^{-1} < E(n) < 2\theta^3\theta'^2$.

We postpone the lengthy proof of Lemma 6 until the next section, and conclude this section by deriving Theorem 2 from Lemma 6. We deal with the three cases in Lemma 6 one at a time.

Case (A). By (23), we have $u(n-2) = u(n) + 1$, so (21) applies. It follows from (21), (25) and (29) that

$$\limsup_{n\to-\infty}\sup_{\alpha}\alpha T_{-1,n}^2\,|R(u(n)-1, n)| = \max_E\,(1 + h_2(E))^2\tau(E),$$

where the max is taken over $-3\theta^2 + 3\theta + 2 \leqq E \leqq E_+$. Calculation shows that the maximum occurs at $E = -3\theta^2 + 3\theta + 2$, and the value of the maximum is $(2\theta + 3)/7 = .78485\ldots$.

Case (B). Suppose first that $2\theta^3\theta'^2 < E(n) < -3\theta^2 + 3\theta + 2$, so $u(n-2) = u(n) + 1$ by (23). Then (21), (25) and (29) imply

$$\limsup_{n\to-\infty}\sup_{\alpha}\alpha T_{-1,n}^2\,|R(u(n)-2, n)| = \max_E\,(1 + h_2(E))^2\theta^{-1}\tau(E),$$

where the max is taken over $2\theta^3\theta'^2 \leqq E \leqq -3\theta^2 + 3\theta + 2$. Calculation shows that the maximum is strictly smaller than $(2\theta + 3)/7$.

Now suppose that $E_- < E(n) < |\theta''|^{-1}$, so $u(n-2) = u(n)$ by (23). Then (19), (25) and (29) imply

$$\limsup_{n\to-\infty}\sup_{\alpha}\alpha T_{-1,n}^2\,|R(u(n)-2, n)| = \max_E\,(1 + h_1(E))^2\theta^{-1}\tau(E),$$

where the max is taken over $E_- \leqq E \leqq |\theta''|^{-1}$. Calculation shows that the maximum occurs at $E = |\theta''|^{-1}$, and the value of the maximum is $(2\theta + 3)/7 = .78485\ldots$.

Case (C). By (23), we have $u(n-2) = u(n)$, so (19), (25) and (29) imply

$$\limsup_{n\to-\infty} \sup_{\alpha} \alpha T^2_{-1,n} |R(u(n-2), n-2)| = \max_{E} (1 + h_1(E))^2 \theta\theta'^2 \tau(E),$$

where the max is taken over $|\theta''|^{-1} \leqq E \leqq 2\theta^3\theta'^2$. Calculation shows that the maximum is less than .3.

Combining the above results for Cases (A), (B) and (C) gives Theorem 2. Indeed, we have proved even more, for the preceding discussion clearly gives considerable information about those values of M for which $\min M^2 |x + \theta y + \theta^2 z|$ in (18) is near its lim sup.

4. Proof of Lemma 6. First we require three computational lemmas.

LEMMA 7. *For each integer m, if $n \to -\infty$ through even values, then $T_{m,n}/T_{-1,n} - q(E(n), m) \to 0$, where*

$$q(E(n), 1) = \frac{-E(n)K'\theta' - K''\theta''}{E(n)G'/\theta' + G''/\theta''} \qquad \text{if } E_- < E(n) < |\theta^3\theta'|^{-1} = 1.1588\ldots,$$

$$q(E(n), 1) = \frac{E(n)G'\theta' + G''\theta''}{E(n)G'/\theta' + G''/\theta''} \qquad \text{if } |\theta^3\theta'|^{-1} < E(n) < E_+,$$

$$q(E(n), 0) = h_2(E(n)),$$

$$q(E(n), -2) = h_1(E(n)) \quad \text{if } E_- < E(n) < |\theta''|^{-1} = .5549\ldots, \tag{31}$$

$$q(E(n), -2) = \frac{-E(n)K'/\theta'^2 - K''/\theta''^2}{E(n)G'/\theta' + G''/\theta''} \quad \text{if } |\theta''|^{-1} < E(n) < E_+, \tag{32}$$

$$q(E(n), m) = \frac{|E(n)K'\theta'^m + K''\theta''^m|}{E(n)G'/\theta' + G''/\theta''} \quad \text{if } m \leqq -3 \text{ or } m \geqq 2. \tag{33}$$

COROLLARY 1. *If $n < 0$, n even and $|n|$ is sufficiently large, then $T_{m,n} > T_{-1,n}$ holds for all integers $m \neq 0, -1$.*

COROLLARY 2. *If $n < 0$, n even and $|n|$ is sufficiently large, then $T_{-1,n} > T_{0,n}$ holds only if $E(n) > |\theta^3\theta'|^{-1} = 1.1588\ldots$.*

COROLLARY 3. *If $n < 0$, n even and $|n|$ is sufficiently large, then for each integer $m \leqq -3$ we have $T^2_{m,n}/T^2_{-1,n} > 7\cdot 4^{-m-3}$.*

Proof. The various statements in (12) determine $T_{m,n}$ for each integer m, with conditions involving the size of $E(n)$ if $m = 1$ or $m = -2$. Thus the formulas for $q(E(n), m)$ follow as in the proof of Lemma 5.

The first corollary follows from the fact that, if $m \neq 0$, $q(E, m) > 1$ holds for all E satisfying $E_- \leqq E \leqq E_+$. The second corollary follows from the fact that $q(E, 0) \geqq 1$ if and only if $E \leqq |\theta^3\theta'|^{-1}$.

To prove the third corollary, we calculate that if $m \leqq -3$, then $q(E, -3)^2 \geqq (2.75)^2 > 7$ and $q(E, m)^2 \geqq (2.75)^2 |\theta'|^{-2m-6} > 7\cdot 4^{-m-3}$ for all E satisfying $E_- \leqq E \leqq E_+$.

LEMMA 8. *If $n \to -\infty$ through even values of n such that $u(n-1) = u(n)$, then*

$$T_{-1,n-1}/T_{-1,n} - (1 + h_1(E(n))) \to 0 \tag{34}$$

and

(35) $$\limsup \frac{T^2_{-1,n-1}}{T^2_{-1,n}} = 6.4376\ldots, \qquad \liminf \frac{T^2_{-1,n-1}}{T^2_{-1,n}} = 4.4707\ldots.$$

If $n \to -\infty$ through even values of n such that $u(n-1) = u(n) + 1$, then

$$T_{-1,n-1}/T_{-1,n} - (1 + h_2(E(n))) \to 0$$

and

(36) $$\limsup \frac{T^2_{-1,n-1}}{T^2_{-1,n}} = 5.5380\ldots, \qquad \liminf \frac{T^2_{-1,n-1}}{T^2_{-1,n}} = 3.5998\ldots.$$

COROLLARY. *If $n < 0$, n even and $|n|$ is sufficiently large, then $T_{-1,n-1} \geqq T_{-1,n-2}$, with equality if and only if $E(n) < 2\theta^3\theta'^2 = .7680\ldots$ or $E(n) > 2|\theta''|^{-1} = 1.1099\ldots$.*

Proof. Note that by Lemma 2 the only possible values for $u(n-1)$ are $u(n)$ and $u(n)+1$. We proceed as in the proof of Lemma 5, using the facts

(37) $$T_{-1,n-1} = |b^{(n)}_{u(n)-1} - g^{(n)}_{u(n)-1} + k^{(n)}_{u(n)-1}| \quad \text{if } u(n-1) = u(n),$$

(38) $$T_{-1,n-1} = |b^{(n)}_{u(n)-1} - g^{(n)}_{u(n)-1} + 2k^{(n)}_{u(n)-1}| \quad \text{if } u(n-1) = u(n)+1,$$

and

(39) $$\begin{aligned} &u(n-1) = u(n) && \text{if and only if } E_- < |E(n)| < 2|\theta''|^{-1} = 1.1099\ldots, \\ &u(n-1) = u(n)+1 && \text{if and only if } 2|\theta''|^{-1} < |E(n)| < E_+. \end{aligned}$$

To prove the corollary, we observe that comparing (37) and (38) with the corresponding formulas for $T_{-1,n-2}$ in the proof of Lemma 5 shows that $T_{-1,n-1} = T_{-1,n-2}$ if $u(n-1) = u(n-2)$; combining (23) and (39) gives the ranges of $E(n)$ for which $u(n-1) = u(n-2)$ holds. For the remaining values of $E(n)$, a calculation using (21) and (34) gives $T_{-1,n-1} > T_{-1,n-2}$.

LEMMA 9. *For each integer m, if $n \to -\infty$ through even values, then*

$$\frac{T_{m,n-1}}{T_{-1,n-1}} - \frac{\big|\,|E(n-1)|\,K'\theta'^m - K''\theta''^m\big|}{|E(n-1)|\,K'/\theta' - K''/\theta''} \to 0.$$

COROLLARY 1. *If $n < 0$, n even and $|n|$ is sufficiently large, then $T_{m,n-1} > T_{-1,n-1}$ holds for all integers $m \neq 0, 1, -1$.*

COROLLARY 2. *If $n < 0$, n even and $|n|$ is sufficiently large, then $T_{-1,n-1} > T_{0,n-1}$ holds only if $|E(n-1)| > \theta^{-1} = .8019\ldots$ and $T_{-1,n-1} > T_{1,n-1}$ holds only if $|E(n-1)| > \theta^2 = 1.5549\ldots$.*

COROLLARY 3. *If $n < 0$, n even and $|n|$ is sufficiently large, then, for each integer $m \leqq -4$, we have $T^2_{m,n-1}/T^2_{-1,n-1} > 20\cdot 4^{-m-4}$.*

Proof. It follows from (12) that $T_{m,n-1} = |k^{(n-1)}_{u(n-1)+m}|$ for all m, so Lemma 9 follows after a by now familiar kind of calculation (note that $E(n-1)$ is negative because $n-1$ is odd).

The proofs of the corollaries parallel the proofs of the similar corollaries to Lemma 7.

We shall prove Lemma 6 by showing that the only possibilities for the minimum in (30) are those given by (A), (B), (C) in the lemma.

The case $j = 0$, $m = -1$. Here (27) reduces to (25), which holds for all sufficiently large $|n|$ by Lemma 5. So the case $j = 0$, $m = -1$ always gives a candidate for the minimum in (30).

The cases $j = 0, m \geqq 0$. For $j = 0, m \geqq 0$, (29) gives

$$T_{-1,n}^2 \; |R(u(n), n)| - \theta^{m+1}\tau(E(n)) \to 0$$

and for $j = 0, m = -1$, (29) gives

$$T_{-1,n}^2 \; |R(u(n) - 1, n)| - \tau(E(n)) \to 0, \tag{40}$$

as $n \to -\infty$ through even values. Since $m \geqq 0$ implies $\tau(E) < \theta^{m+1}\tau(E)$ for any E, the min in (30) cannot occur for $j = 0, m \geqq 0$ if $|n|$ is sufficiently large.

The case $j = 0, m = -2$. In this case (29) gives

$$T_{-1,n}^2 \; |R(u(n) - 2, n)| - \theta^{-1}\tau(E(n)) \to 0. \tag{41}$$

The corresponding result for $j = 0, m = -1$ is (40), so $j = 0, m = -2$ is a better candidate for the min in (30) whenever (27) holds for $j = 0, m = -2$. Calculations using Lemma 5 and (31), (32) of Lemma 7 show that, for sufficiently large $|n|$, $T_{-1,n-2}/T_{-1,n} > T_{-2,n}/T_{-1,n}$ is true if $E(n) < -3\theta^2 + 3\theta + 2 = 1.0760 \ldots$, but false if $E(n) > -3\theta^2 + 3\theta + 2$. Thus $j = 0, m = -2$ is a better candidate than $j = 0, m = -1$ if and only if $E(n) < -3\theta^2 + 3\theta + 2$.

The cases $j = 0, m \leqq -3$. For $m \leqq -3$, the size of $T_{m,n}^2/T_{-1,n}^2$ is determined by Lemma 7, (33); calculation shows that $q(E, m)^2 \geqq (2.75)^2 > 7$ for $E_- \leqq E \leqq E_+$. Since $T_{-1,n-2}^2/T_{-1,n}^2 \leqq 6$ for $|n|$ large by (20) and (22), we find that (27) never holds for $j = 0, m \leqq -3$, $|n|$ large.

The case $j = -2$. By Lemma 7, Corollary 1, (27) with $j = -2$ is possible for large $|n|$ only if $m = 0$. Even when $m = 0$, by Lemma 7, Corollary 2, (27) holds only if $E(n - 2) > |\theta^3\theta'|^{-1} = 1.1588 \ldots$.

By (23), $E(n - 2) > 1.1588$ implies $u(n - 4) = u(n - 2) + 1$, and by Lemma 2 this implies $u(n - 2) = u(n)$. Thus we have (see the discussion below (23)) $E(n - 2) = E(n)/\theta^4\theta'^2$, and (27) holds only if $2\theta^3\theta'^2 = .7680 \ldots > E(n) > |\theta\theta'| = |\theta''|^{-1} = .5549 \ldots$. For these values of $E(n)$, $j = -2, m = 0$ is a better candidate for the min in (30) than $j = 0, m = -2$, because (29) gives

$$T_{-1,n}^2 \; |R(u(n - 2), n - 2)| - \theta\theta'^2\tau(E(n)) \to 0$$

if $j = -2, m = 0$, and the constant $\theta\theta'^2 = .2469 \ldots$ is smaller than the constant $\theta^{-1} = .8019 \ldots$ in the corresponding formula (41) for the case $j = 0, m = -2$.

Putting together the candidates j, m for the min in (30) which we have so far, we obtain exactly the Cases (A), (B), (C) of Lemma 6. Thus the proof of Lemma 6 is complete if we show that the pairs j, m not yet considered (that is, pairs with $j \neq 0$ and $j \neq -2$) never give the minimum in (30).

The cases $j \leqq -4$, *j even.* It follows from (27) that $T_{-1,n-2} > T_{m,n+j}$. We know from Lemma 5 that $T_{-1,n-2} < T_{-1,n-4}$ for all large $|n|$ and from Lemma 7, Corollaries 1 and 2, that $T_{-1,n-4} < T_{m,n-4}$ for $m \neq 0$ or $E(n - 4) < |\theta^3\theta'|^{-1}$, and all large $|n|$. Hence if $j \leqq -4$, j even, (27) is possible with $|n|$ large only if $T_{-1,n} > T_{0,n-2}$ can occur with $|n|$ large and $E(n - 2) > |\theta^3\theta'|^{-1}$. As we saw in the case $j = -2$, this implies $u(n - 2) = u(n)$, so the reasoning of the proof of Lemma 5 gives

$$T_{0,n-2} = |b_{u(n)-1}^{(n)} - g_{u(n)-1}^{(n)} + 2k_{u(n)-1}^{(n)}|.$$

As in the proof of (21), we find that $T_{0,n-2}/T_{-1,n} - (1 + h_2(E(n))) \to 0$ as $n \to -\infty$ through even values. The function $1 + h_2(E)$ always exceeds 1, so $T_{-1,n} > T_{0,n-2}$

cannot occur for large $|n|$. This eliminates the cases $j \leqq -4$, j even as candidates for the min in (30).

The cases $j \geqq 2$, j even. Let $j = 2t$, $t \geqq 1$; then the following lower bound on the coefficient of $\tau(E(n))$ in (29) is a trivial consequence of Lemma 2 and the fact that $|\theta'|^{-2} = 5.0489 \ldots > 5$:

$$\theta^{u(n+j)-u(n)+m+1} \, |\theta'|^{-j} \geqq \theta^{m+1-t} \, |\theta'|^{-2t} > \theta^{m+1-t} 5^t.$$

The coefficient of $\tau(E(n))$ in (29) is 1 if $j = 0$, $m = -1$ (see (40)); hence an integer pair $j = 2t$, m can be a candidate for the minimum in (30) only if

$$\theta^{m+1-t} 5^t < 1. \tag{42}$$

Since $t \geqq 1$, (42) implies $m \leqq -3$.

If (27) holds for $j = 2t$, $t \geqq 1$, $m \leqq -3$, we deduce the following for all sufficiently large $|n|$:

$$6^{t+1} T^2_{-1,n+j} \geqq T^2_{-1,n-2} > T^2_{m,n+j} > 7 \cdot 4^{-m-3} T^2_{-1,n+j}. \tag{43}$$

The first inequality follows from (20) and (22), the second from (27), and the third from Lemma 7, Corollary 3.

Now (43) implies $6^{t+1} > 7 \cdot 4^{-m-3}$, which gives a lower bound for m in terms of t. Calculation shows that for any $t \geqq 1$, this lower bound contradicts (42). Hence the cases $j \geqq 2$, j even never give a candidate for the min in (30).

The case $j = -1$. If $j = -1$ and $m \geqq 2$ or $m \leqq -1$, it follows immediately from Lemma 8, Corollary and Lemma 9, Corollary 1 that (27) is impossible for large $|n|$.

If $j = -1$, $m = 0$, it follows from Lemma 8, Corollary and Lemma 9, Corollary 2 that (27) is possible only if $|E(n-1)| > \theta^{-1} = .8019 \ldots$; this inequality implies $E(n-1) = E(n)/\theta^2\theta'$, and so is equivalent to $2\,|\theta\theta'| = 1.1099 \ldots > E(n) > |\theta\theta'| = |\theta''|^{-1} = .5549 \ldots$. In the range $2\theta^3\theta'^2 = .7680 \ldots > E(n) > |\theta''|^{-1}$, the candidate $j = -2$, $m = 0$ for the min in (30) is superior to the candidate $j = -1$, $m = 0$, because the coefficient of $\tau(E(n))$ in (29) is $\theta\theta'^2 = .2469 \ldots$ if $j = -2$, $m = 0$ and is $|\theta\theta'| = .5549 \ldots$ if $j = -1$, $m = 0$. In the remaining range $2\,|\theta\theta'| > E(n) > 2\theta^3\theta'^2$, we have for $|n|$ large

$$\frac{T_{-1,n-1}}{T_{-1,n}} \cdot \frac{T_{0,n-1}}{T_{-1,n-1}} = \frac{T_{0,n-1}}{T_{-1,n}} > (5.6)(.85) > 4.7 > \frac{T_{-1,n-2}}{T_{-1,n}} \tag{44}$$

(the first inequality follows from a calculation using (34) and Lemma 9 with $m = 0$, and the third inequality follows from Lemma 5, (22)), so (27) does not hold. This completes the proof that $j = -1$, $m = 0$ does not give the minimum in (30).

If $j = -1$, $m = 1$, it follows from Lemma 8, Corollary and Lemma 9, Corollary 2 that (27) is possible only if $|E(n-1)| > \theta^2 = 1.5549 \ldots$, that is, only if $2\,|\theta\theta'| = 1.1099 \ldots > E(n) > |\theta^4\theta'| = 1.076 \ldots$; but (44) shows that (27) does not hold for this range of $E(n)$.

The cases $j \leqq -3$, j odd. If (27) is true for $j \leqq -3$, j odd, and any m, then $T_{-1,n-2} > T_{m,n+j}$. However, for large $|n|$ a calculation using Lemma 9 gives $T^2_{m,n+j} > .6T^2_{-1,n+j}$ for any m and (35), (36) give $T^2_{-1,n+j} > 3.5T^2_{-1,n+j+1}$. Thus (27) would imply $T^2_{-1,n-2} > 2.1T^2_{-1,n+j+1}$, which plainly contradicts Lemma 5 for $j \leqq -3$, j odd.

The cases $j \geqq 1$, j odd. Let $j = 2t + 1$, $t \geqq 0$; these cases are dealt with in the same way as the cases $j \geqq 2$, j even above.

First we find a trivial lower bound on the coefficient of $\tau(E(n))$ in (29):

$$\theta^{u(n+j)-u(n)+m+1} |\theta'|^{-j} \geqq \theta^{m-t} |\theta'|^{-2t-1} > 2\theta^{m-t}5^t.$$

The analogue of (42) is

$$(45) \qquad 2\theta^{m-t}5^t < 1$$

(which implies $m \leqq -4$ since $t \geqq 0$) and the analogue of (43) is

$$(46) \qquad 6^{t+1}T^2_{-1,n+j} \geqq T^2_{-1,n-2} > T^2_{m,n+j} > 20 \cdot 4^{-m-4} T^2_{-1,n+j},$$

where the last inequality follows from Lemma 9, Corollary 3. Calculation shows that, for any $t \geqq 0$, the bounds on m given by (45) and (46) are contradictory.

All cases have now been covered, so the proof of Lemma 6 is complete.

5. Concluding Remarks. It is clear that if the method of [2] can be applied to a linear form $x + \alpha y + \beta z$, then the method of the present paper can be used to attempt to find $c_2(\alpha, \beta)$. The attempt will succeed if and only if the method of [2] gives *all* solutions of $|x + \alpha y + \beta z| \max(y^2, z^2) < c$ for a *sufficiently large* c. It is certainly sufficient if $c \geqq 1$, as was the case for the example in this paper (see (4) and the first paragraph after (25)). It may even be the case that if the method of [2] applies to the linear form $x + \alpha y + \beta z$, then the constant c can always be taken large enough to permit the evaluation of $c_2(\alpha, \beta)$; but it appears to be difficult to prove anything in this direction.

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