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Accurate Evaluation of Wiener Integrals*

By Alexandre Joel Chorin

Abstract. A new quadrature formula for an important class of Wiener integrals is presented, in which the Wiener integrals are approximated by n-fold integrals with an error $O(n^{-2})$. The resulting n-fold integrals can then be approximated by ordinary finite sums of remarkably simple structure. An example is given.

Introduction. Wiener integrals in function space play a major role in a number of applications in physics and in probability theory, see e.g. [1], [6], [7], [9]. A number of remarkable results have been obtained concerning the approximation of these integrals by finite-dimensional integrals (see in particular Cameron [2], as well as [8], [10], and [14]). The resulting n-fold integrals are, in general, difficult to evaluate with any accuracy, and as a consequence the approximation formulas are not of significant practical use. The aim of this paper is to present a new approximation for Wiener integrals accurate enough and simple enough to be of practical interest. Some of the elegant generality of Cameron's work may be lost, but the method is applicable to many functionals which appear in physics, and will furthermore afford an intuitive grasp of the relation between ordinary quadrature and quadrature in a function space.

The two main ideas in the approximation method are the following: the Wiener paths are carefully interpolated by a certain family of parabolas, in such a way that all the moments are exactly reproduced; and nonlinear functionals are expanded in a certain Taylor series, with the quadrature formula adjusted so that the first two groups of terms are well approximated.

Outline of Goal and Method. Let C be the space of continuous real functions $x(t)$ defined on $0 \leqq t \leqq 1$, with $x(0) = 0$, and endowed with the Wiener measure W. Let $F[x]$ be a functional on C; our aim is to evaluate

$$J = \int_C F[x]\, dW;$$

we shall construct approximation formulas of the form

$$\begin{aligned}(1)\qquad \int_C F[x]\, dW = \pi^{-n/2} \int_{R^n} & F_n(u_1, u_2, \cdots, u_n) \\ & \cdot \exp(-u_1^2 - u_2^2 - \cdots - u_n^2)\, du_1\, du_2 \cdots du_n \\ & + O(n^{-2})\end{aligned}$$

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where it is required that the n-fold integral be capable of approximation by an ordinary finite sum of simple structure without increasing the order of magnitude of the error.

In general, the variables $u_1, \cdots, u_n$ will be linear functionals of the paths $x(t)$; our quadrature formula will thus be based on an approximation of a functional of the paths by a function of linear functionals. We shall have

$$u_i = \int_0^1 \alpha_i(t)\, dx(t), \qquad i = 1, \cdots, n, \tag{2}$$

where the α_i are ordinary functions on $0 \leqq t \leqq 1$ satisfying

$$\int_0^1 \alpha_i(t)\alpha_j(t)\, dt = \delta_{i,j}, \qquad \delta_{i,j} \text{ the Kronecker delta.}$$

The integrals (2) exist as generalized Stieltjes integrals for almost all $x(t)$, see [12]. The resulting u_i are independent, gaussianly distributed random variables with mean 0 and variance $1/2$ ([12], [15]). They play a major role in the construction of Wiener-Hermite polynomials [3]. In intuitive terms, the $\alpha_i(t)$ should be chosen in such a way that the finite-dimensional space spanned by linear combinations of their derivatives contain most of the information required for the evaluation of the integral. If no information about F is available, then there is no rational basis for making a choice and one may follow the example of Cameron [2] and fix the $\alpha_i(t)$ in advance. Sometimes, there exists a natural choice: for example, the solution of the one-dimensional heat equation

$$v_t = \tfrac{1}{4}\, v_{xx}, \qquad v(x, 0) \text{ given,}$$

can be written as a Wiener integral. This Wiener integral is equal to a one-dimensional integral of a function of

$$u_1 = \int_0^1 1 \cdot dx(t) = x(1);$$

this integral is merely the Green's function representation of the solution (see [4]).

In the following sections, we shall construct quadrature formulas for functionals of the form

$$F[x] = g(x(t_1), x(t_2), \cdots, x(t_m))G\left(\int_0^1 V(x(t))\, dt\right),$$

where $G(y)$, $V(y)$ are ordinary functions of their arguments, $g(y_1, \cdots, y_m)$ is an ordinary function of m variables, and $t_1, t_2, \cdots, t_m$ are fixed values of t. A case of major importance in physics is $G(y) = \exp(-y)$. We shall begin by constructing quadrature formulas of arbitrary accuracy for some special functionals, and then proceed to the more general case.

Integrals of Some Special Functionals. Consider first the functionals

$$F[x] = \int_0^1 x^m(t)\, dt, \ m \text{ integer.} \tag{3}$$

Their integrals can be readily evaluated; we have

$$
\begin{aligned}
(4)\qquad \int_C \left\{\int_0^1 x^m(t)\,dt\right\} dW &= \int_0^1 \left\{\int_C x^m(t)\,dW\right\} dt \\
&= \int_0^1 C_m t^{m/2}\,dt = C_m/(1 + m/2),
\end{aligned}
$$

where

$$
\begin{aligned}
C_m &= 0, && m \text{ odd}, \\
&= 2^{-m/2}(m-1)(m-3)\cdots(1), && m \text{ even}.
\end{aligned}
$$

The change in the order of integration can be justified by application of Fubini's theorem (see [2]).

We now construct a quadrature rule which yields exactly the result (4) for all m. Set

$$F_1(v) = \int_0^1 (\sqrt{t}\,v)^m\,dt,$$

i. e., evaluate the functional (3) on the special paths $x(t) = \sqrt{t}\,v$. One can readily verify that

$$\pi^{-1/2} \int \cdot \left\{\int_0^1 (\sqrt{t}\,v)^m\,dt\right\} e^{-v^2}\,dv = C_m/(1 + m/2),$$

in exact agreement with (4) for all m; i. e., the moments of $x(t)$ are reproduced exactly.

Now note that if the integration in t is approximated by a quadrature rule which yields an exact answer for all polynomials of degree less than or equal to $[m/2]$, where $[m/2]$ denotes the integer part of $m/2$, and if the integration in v is approximated by a weighted Gaussian quadrature formula which yields an exact answer for all integrals of the form

$$\int v^{m'} e^{-v^2}\,dv, \qquad m' \leqq m,$$

(such quadrature formulas are given e. g. in [13]), then the resulting ordinary finite sum will still yield the exact value of the integral of the functional (3). The important point is that the half-integer powers of t, for which the quadrature rule yields an inaccurate answer, are multiplied by odd powers of v, and thus, after integration with respect to v, do not affect the answer. Now consider functionals of the form

$$F[x] = \int_0^1 V(x(t))\,dt,$$

where $V(y)$ is an ordinary function of the real argument y, having m derivatives V', V'', $\cdots$, $V^{(m)}$ with, for all y, y_0,

$$
\begin{aligned}
V(y_0 + y) = V(y_0) + V'(y_0)y + \cdots &+ \frac{1}{(m-1)!} V^{(m-1)}(y_0)y^{m-1} \\
&+ \frac{1}{m!} V^{(m)}(y_0 + \theta y)y^m, \qquad 0 \leqq \theta = \theta(y) \leqq 1,
\end{aligned}
$$

where

$$\int_C \left\{ \int_0^{\Delta} V^{(m)}(y_0 + \theta(x(t))x(t))x^m(t)\, dt \right\} dW = O(\Delta^{m/2+1}) \quad \text{as } \Delta \to 0 \text{ and for all } y_0.$$

Divide the interval $0 \leqq t \leqq 1$ into n intervals $I_1, I_2, \cdots, I_n$ of equal lengths n^{-1}; define

$$\alpha_i(t) = \sqrt{n}, \qquad t \in I_i,$$
$$= 0, \qquad t \notin I_i,$$

and

$$u_i = \int_0^1 \alpha_i(t)\, dx(t).$$

We note that the derivatives of the $\alpha_i(t)$ are delta functions. Write

$$x_i = x(i/n) \qquad (x_0 = 0),$$
$$V_i = V(x_i), \qquad V_i' = V'(x_i), \quad \text{etc.},$$
$$\Delta x_i = x(t) - x((i-1)/n), \qquad (i-1)/n \leqq t \leqq i/n;$$

we have $u_i = (x_i - x_{i-1}) \cdot \sqrt{n}$, and, conversely, $x_i = (1/\sqrt{n})(u_1 + \cdots + u_i)$. We can write

$$\int_C F[x]\, dW \equiv \int_C \left\{ \int_0^1 V(x(t))\, dt \right\} dW$$
$$= \int_C \left\{ \sum_{i=1}^n \int_{(i-1)/n}^{i/n} V(x(t))\, dt \right\} dW = \int_C \left\{ \sum_{i=1}^n \int_0^{1/n} V(x_{i-1} + \Delta x_i(t))\, dt \right\} dW$$
$$= \int_C \left\{ \sum_{i=1}^n \int_0^{1/n} (V_{i-1} + V_{i-1}' \Delta x_i + \cdots + V_{i-1}^{(m-1)} \cdot (\Delta x_i)^{m-1}) \right\} dW + O(n^{-m/2}),$$

i. e.,

$$\int_C F[x]\, dW = \pi^{-n/2} \int \left\{ \sum_{i=0}^n \int_0^{1/n} V(x_{i-1} + \sqrt{t}\, v)\, dt \right\}$$
$$\tag{5} \cdot \exp(-u_1^2 - u_2^2 - \cdots - u_{n-1}^2 - v^2)$$
$$\cdot du_1\, du_2 \cdots du_{n-1}\, dv + O(n^{-m/2}),$$
$$x_{i-1} = \frac{1}{\sqrt{n}}(u_1 + \cdots + u_{i-1}).$$

(5) is of the form (1), and, as before, the integrals can be approximated by means of finite sums.

Two remarks remain to be made, for use in the next section. The approximation (5) can be derived through the use of the interpolation formula for Wiener paths [11]: if $x((i-1)/n) = x_{i-1}$, $x(i/n) = x_{i-1} + u_i/\sqrt{n}$, then for t such that $(i-1)/n \leqq t \leqq i/n$ we have

$$\tag{6} x(t) = x_{i-1} + u_i \sqrt{n}\, \Delta t + w_i \sqrt{n} \left(\Delta t \left(\frac{1}{n} - \Delta t \right) \right)^{1/2}, \quad \Delta t = t - (i-1)/n,$$

where w_i is a gaussianly distributed random variable with mean 0 and variance $\frac{1}{2}$. Thus,

$$
\begin{aligned}
\int_C F[x]\,dW &= \int_C \left\{\sum_{i=1}^{n} \int_0^{1/n} V(x_{i-1} + \Delta x_i)\,dt\right\} dW \\
&= \pi^{-n} \int \left\{\sum_{i=1}^{n} \int_0^{1/n} V\left(x_{i-1} + u_i\sqrt{n}\,t + w_i\sqrt{n}\left(t\left(\frac{1}{n} - t\right)\right)^{1/2} dt\right)\right\} \\
&\quad \cdot \exp(-u_1^2 - u_2^2 - \cdots - u_n^2 - w_1^2 - w_2^2 - \cdots - w_n^2) \\
&\quad \cdot du_1 \cdots du_n\, dw_1 \cdots dw_n + O(n^{-m/2}).
\end{aligned} \tag{7}
$$

Some elementary algebra yields

$$
\begin{aligned}
\pi^{-1} \int \left\{\int_0^{1/n} \left(u_i\sqrt{n}\,t + w_i\sqrt{n}\left(t\left(\frac{1}{n} - t\right)\right)^{1/2}\right)^m dt\right\} \exp(-u_i^2 - w_i^2)\,du_i\,dw_i \\
= \pi^{-1/2} \int \left\{\int_0^{1/n} (\sqrt{t}\,v)^m\,dt\right\} \exp(-v^2)\,dv
\end{aligned}
$$

for all m, and thus the $(2n + 1)$-fold integral (7) and the $(n + 1)$-fold integral (5) are always equal. This is of course true only because the functional under consideration is linear in the partial integrals $\int_0^{1/n} V(x_i(t))\,dt$. Finally, if one is content with accuracy of order $O(n^{-2})$, one may replace the t-integration by a one-term midpoint rule, i. e., use

$$
\int_0^{1/n} t\,dt = \frac{1}{n}\cdot\frac{1}{2n}, \qquad \int_0^{1/n} t^2\,dt = O(n^{-3}),
$$

to obtain

$$
\begin{aligned}
\int_C F[x]\,dW = \pi^{-n/2} \int \left\{\sum_{i=1}^{n} \frac{1}{n} V(x_{i-1} + v/(2n)^{1/2})\right\} \\
\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\,du_1 \cdots du_{n-1}\,dv \\
+ O(n^{-2}).
\end{aligned} \tag{8}
$$

Functionals which are Functions of an Integral. In this section, we consider functionals of the form

$$
F[x] = G\left(\int_0^1 V(x(t))\,dt\right),
$$

where $V = V(y)$ is a function of y satisfying the conditions above, and $G(y)$ is a sufficiently smooth function of the real argument y. The precise requirements on G will appear below. The main result of this section is a remarkably simple formula, of which (8) is a special case:

$$
\begin{aligned}
\int_C G\left(\int_0^1 V(x(t))\,dt\right) dW = \pi^{-n/2} \int \left\{G\left(\sum_{i=1}^{n} \frac{1}{n} V(x_{i-1} + v/(2n)^{1/2})\right)\right\} \\
\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2) \\
\cdot du_1 \cdots du_{n-1}\,dv + O(n^{-2}).
\end{aligned} \tag{9}
$$

As in the previous section, we divide the interval $0 \leqq t \leqq 1$ into n subintervals of length n^{-1}, and define $x_i, u_i, V_i, \Delta x_i, i = 1, \cdots, n$, as above. Note that the variables $\Delta x_i, \Delta x_j, i \neq j$, are independent, by definition of the Wiener process, as are the variables $u_i, u_j, i \neq j$, and $\Delta x_i, u_j, i \neq j$. Of course, $u_i, \Delta x_i$ are correlated.

We introduce the following notations

$$q_i = \frac{1}{n} V_{i-1}, \qquad \Delta q_i = \int_0^{1/n} [V(x_{i-1} + \Delta x_i) - V_{i-1}]\, dt,$$

thus

$$\int_{(i-1)/n}^{i/n} V(x(t))\, dt = q_i + \Delta q_i$$

and

$$\begin{aligned} F[x] &= G\left(\int_0^1 V(x(t))\, dt\right) = G\left(\sum_{i=1}^n \int_0^{1/n} V(x_{i-1} + \Delta x_i)\, dt\right) \\ &= G\left(\sum_{i=1}^n (q_i + \Delta q_i)\right) \\ &= G\left(\sum_{i=1}^n q_i\right) + G'\left(\sum_{i=1}^n q_i\right) \sum_{j=1}^n \Delta q_j \\ &\quad + \frac{1}{2} G''\left(\sum_{i=1}^n q_i\right) \sum_{j=1}^n \sum_{k=1}^n \Delta q_i\, \Delta q_j \\ &\quad + \frac{1}{6} G'''\left(\sum q_i\right) \sum_j \sum_k \sum_l \Delta q_j\, \Delta q_k\, \Delta q_l \\ &\quad + \frac{1}{24} G^{\text{iv}}_{jklm} \sum_j \sum_k \sum_l \sum_m \Delta q_j\, \Delta q_k\, \Delta q_l\, \Delta q_m, \end{aligned} \tag{10}$$

where $G', G'', G''', G^{\text{iv}}$ are the derivatives of G with respect to y, which are assumed to exist, and

$$G^{\text{iv}}_{jklm} = G^{\text{iv}}\left(\sum_{i=1}^n q_i + \theta_j \Delta q_j + \theta_k \Delta q_k + \theta_l \Delta q_l + \theta_m \Delta q_m\right), \quad 0 \leqq \theta_j, \theta_k, \theta_l, \theta_m \leqq 1.$$

We first show that the contribution of the last three sums to the integral of $F[x]$ is of order n^{-2}. We have

$$\left| \int_C G^{\text{iv}}_{jklm} \Delta q_j \Delta q_k \Delta q_l \Delta q_m\, dW \right| \leqq I^{1/2} \left\{ \int_C \Delta q_j^2 \Delta q_k^2 \Delta q_l^2 \Delta q_m^2\, dW \right\}^{1/2},$$

where $I = \int_C (G^{\text{iv}}_{jklm})^2\, dW$ is assumed bounded for all j, k, l, m. Furthermore,

$$\begin{aligned} &\left\{ \int_C \Delta q_j^2 \Delta q_k^2 \Delta q_l^2 \Delta q_m^2\, dW \right\}^{1/2} \\ &\qquad \leqq \left(\int \Delta q_j^8\, dW\right)^{1/8} \left(\int \Delta q_k^8\, dW\right)^{1/8} \left(\int \Delta q_l^8\, dW\right)^{1/8} \left(\int \Delta q_m^8\, dW\right)^{1/8}; \end{aligned}$$

by definition,

$$\Delta q_j = \int_0^{1/n} [V'_{j-1}\Delta x_j + \tfrac{1}{2} V''(x_{j-1} + \theta\Delta x_j)\Delta x_j]\, dt, \qquad 0 \leqq \theta \leqq 1,$$

and, therefore, if expressions such as

$$\left\{\int_C \frac{1}{n}(V'_{j-1})^{2\alpha}\left(\int_0^{1/n} V''(x_{j-1} + \theta\Delta x_j)\, dt\right)^{2\beta} dW\right\}^{1/2},$$

α, β integers, $\alpha + \beta = 8$, are bounded, we have

$$\int \Delta q_j^8\, dW = O(n^{-12}), \qquad \left(\int \Delta q_j^8\, dW\right)^{1/8} = O(n^{-3/2}),$$

and a typical term in the last sum in (10) is $O(n^{-6})$. There are n^4 terms in the last sum in (10), and thus, the total contribution is $O(n^{-2})$.

Before considering the other terms, we introduce the notations

$$\begin{aligned} V_i^{(j)} &= V_i = V(x_i), & i > j,\\ &= V\left(x_i - \frac{u_j}{\sqrt{n}}\right), & i \leqq j, \end{aligned}$$

and

$$\begin{aligned} V_j^{(ik)} &= V_j = V(x_j), & j < i, j < k,\\ &= V\left(x_j - \frac{u_i}{\sqrt{n}}\right), & i \leqq j < k,\\ &= V\left(x_j - \frac{u_k}{\sqrt{n}}\right), & k \leqq j < i,\\ &= V\left(x_j - \frac{u_i}{\sqrt{n}} - \frac{u_k}{\sqrt{n}}\right), & i \leqq j, k \leqq j, \text{ etc.}, \end{aligned}$$

i. e., we write in superscript the indices of those among the variables $u_1, \cdots, u_n$ which we set equal to zero in the argument of V. Thus,

$$G'''\left(\sum_{i=1}^n q_i\right) = G'''\left(\sum_{i=1}^n \frac{1}{n} V_{i-1}^{(jkl)}\right) + G^{\mathrm{iv}}\left(\sum_{i=1}^n \frac{1}{n} V(\alpha_i)\right)\frac{u_j}{\sqrt{n}} \tag{11}$$
$$+ \text{ two similar terms in } u_k, u_l,$$

where

$$\begin{aligned} \alpha_i &= x_{i-1}, \qquad i < j,\\ &= (1/\sqrt{n})(u_1 + u_2 + \cdots + u_{j-1} + \theta u_j + u_{j+1} + \cdots + u_{i-1}),\\ &\qquad\qquad i \geqq j,\ 0 \leqq \theta \leqq 1. \end{aligned}$$

Furthermore,

$$\Delta q_j = \int V'_{j-1}\Delta x_j\, dt + \frac{1}{2}\int V''_{j-1}\cdot(\Delta x_j)^2\, dt + \cdots . \tag{12}$$

Now, if any one of the indices j, k, l is larger than the others, for example $l > j, l > k$, then

$$\int_C \left\{ \int_0^{1/n} V'_{j-1}\Delta x_j \, dt \int_0^{1/n} V'_{k-1}\Delta x_k \, dt \int_0^{1/n} V'_{l-1}\Delta x_l \, dt \right\} dW = 0,$$

since the expression in curly brackets is an odd function of the random variable Δx_l. For the same reason,

$$\int_C \left\{ G'''\left(\sum_{i=1}^n \frac{1}{n} V_{i-1}^{(jkl)} \right) \int V'_{j-1}\Delta x_j \, dt \int V'_{k-1}\Delta x_k \, dt \int V'_{l-1}\Delta x_l \, dt \right\} dW = 0.$$

An inspection of (11) and (12) shows that for $l > j$ and $l > k$, or $j > l$ and $j > k$, or $k > l$ and $k > j$,

$$\int \left\{ G'''\left(\sum q_i \right) \Delta q_j \Delta q_k \Delta q_l \right\} dW = O(n^{-5}),$$

there are $O(n^3)$ such terms, and their total contribution is thus $O(n^{-2})$. There remain terms for which $j = k > l$, or $j = l > k$, or $l = k > j$; they are of order $n^{-9/2}$ but there are only $O(n^2)$ such terms. A similar analysis shows that

$$\int \left\{ G''\left(\sum_{i=1}^n q_i \right) \sum_j \sum_k \Delta q_j \Delta q_k \right\} dW = O(n^{-2}),$$

and thus

$$\int F[x] \, dW = \int \left\{ G\left(\sum_{i=1}^n q_i \right) + G'\left(\sum_{i=1}^n q_i \right) \sum_{j=1}^n \Delta q_j \right\} dW + O(n^{-2}). \tag{12}$$

This is our main formula; it shows that accuracy of order n^{-2} can be obtained provided the finite-dimensional integral reproduces the integral of the first terms in the Taylor expansion with sufficient accuracy. The crucial fact is that those first terms are linear in the Δq_j. To evaluate the integral on the right-hand side of (12), we make use of the interpolation formula (6). We integrate the functional F over all paths such that

$$(1/\sqrt{n})(u_1 + u_2 + \cdots + u_i) \leqq x(i/n)$$
$$\leqq (1/\sqrt{n})(u_1 + u_2 + \cdots + u_i) + (1/\sqrt{n})(du_1 + \cdots + du_i)$$

and then integrate over all values of $u_1, u_2, \cdots, u_n$. This yields

$$\int \left\{ G\left(\sum_{i=1}^n q_i \right) + G'\left(\sum_{i=1}^n q_i \right) \sum_{j=1}^n \Delta q_j \right\} dW$$

$$\begin{aligned} &= \pi^{-n} \int \left\{ G\left(\sum_{i=1}^n \frac{1}{n} V(x_{i-1}) \right) + G'\left(\sum_{i=1}^n \frac{1}{n} V(x_{i-1}) \right) \right. \\ &\quad \left. \cdot \sum_{j=1}^n \int_0^{1/n} \left(V\left(x_{j-1} + u_j \sqrt{n}\, t + w_j \sqrt{n} \left(t\left(\frac{1}{n} - t \right) \right)^{1/2} \right) - V(x_{j-1}) \right) dt \right\} \\ &\quad \cdot \exp(-u_1^2 - \cdots - u_n^2 - w_1^2 - \cdots - w_n^2) \, du_1 \cdots du_n \, dw_1 \cdots dw_n \end{aligned} \tag{13}$$

which can be seen to differ only by $O(n^{-2})$ from

$$\begin{aligned} &\pi^{-n} \int \left\{ G\left(\sum_{i=1}^n \int_0^{1/n} V\left(x_{i-1} + u_i \sqrt{n}\, t + w_i \sqrt{n} \left(t\left(\frac{1}{n} - t \right) \right)^{1/2} \right) dt \right) \right\} \\ &\quad \cdot \exp(-u_1^2 - \cdots - u_n^2 - w_1^2 - \cdots - w_n^2) \, du_1 \cdots du_n \, dw_1 \cdots dw_n. \end{aligned} \tag{14}$$

The $(2n + 1)$-fold integral (14), an analogue of (7), approximates the integral of our functional with an error of order n^{-2}. We now proceed to simplify formula (14).

Define

$$\Delta X_i = u_i \sqrt{n}\, t + w_i \sqrt{n}\, (t(1/n - t))^{1/2},$$

then

$$(15) \qquad \int_0^{1/n} \Delta X_i \, dt = u_i \sqrt{n} \frac{1}{2n^2} + w_i \sqrt{n} \int_0^{1/n} \left(t\left(\frac{1}{n} - t\right)\right)^{1/2} dt,$$

$$(16) \qquad \int_0^{1/n} (\Delta X_i)^2 \, dt = u_i^2 \frac{1}{3n^2} + w_i^2 \frac{1}{6n^2} + 2u_i w_i n \int_0^{1/n} t\left(t\left(\frac{1}{n} - t\right)\right)^{1/2} dt,$$

$$(17) \qquad \int_0^{1/n} (\Delta X_i)^3 \, dt = u_i^3 \cdot O(n^{-5/2}) + w_i^3 \cdot O(n^{-5/2}) + u_i^2 w_i \cdot O(n^{-5/2}) + u_i w_i^2 \cdot O(n^{-5/2}).$$

Consider a typical term in the sum on the right-hand side of Eq. (13), for example

$$(18) \qquad G'\left(\sum_{i=1}^{n} \frac{1}{n} V(x_{i-1})\right) \cdot \int_0^{1/n} \left(V\left(x_{j-1} + u_j \sqrt{n}\, t + w_j \sqrt{n} \left(t\left(\frac{1}{n} - t\right)\right)^{1/2}\right) - V(x_{j-1})\right) dt.$$

The variables Δx_j, u_j are correlated, and this correlation manifests itself through the presence of the variable u_j in both terms of the product. Let us put the role of u_j in evidence. Write

$$(19) \qquad v = u_j.$$

Since for $j > i$,

$$(1/\sqrt{n})(u_1 + u_2 + \cdots + u_{j-1} + v + u_{j+1} + \cdots + u_{i-1}) = x_{i-1} + (v - u_j)/\sqrt{n},$$

the change of variable (19) changes G' into

$$G'\left(\sum_{i=1}^{j} \frac{1}{n} V_{i-1} + \sum_{i=j+1}^{n} \frac{1}{n} V\left(x_{i-1} + \frac{v - u_j}{\sqrt{n}}\right)\right)$$
$$= G'\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}\right) + G''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}\right) \sum_{i=j+1}^{n} V'(x_{i-1}) \frac{v - u_j}{\sqrt{n}} + \cdots,$$

where the three dots denote the obvious remainder term. Furthermore,

$$G''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}\right) V'(x_{i-1}) \frac{u_j}{n}$$
$$= G''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right) V'(x_{i-1}) \frac{u_j}{n}$$
$$+ \frac{1}{2}\left\{G'''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right) \frac{1}{n} \sum_{i=j+1}^{n} V'^{(j)}(x_{i-1}) V'(x_{i-1})\right\} \frac{u_j^2}{n^{3/2}} + \cdots .$$

On the other hand, using (15), (16) and (17), we have

$$\int_0^{1/n}\left(V\left(x_{j-1}+v\sqrt{n}\,t+w_j\sqrt{n}\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}\right)-V(x_{j-1})\right)dt$$
$$=V'_{j-1}\cdot\left(v\sqrt{n}\,\frac{1}{2n^2}+w_i\sqrt{n}\int_0^{1/n}\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}dt\right)$$
$$+\frac{1}{2}V''_{j-1}\cdot\left(v^2\frac{1}{3n^2}+w_\iota^2\frac{1}{6n^2}+2u_\iota w_\iota n\int_0^{1/n}t\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}dt\right)+\cdots.$$

Carrying out the multiplications, dropping all terms $O(n^{-3})$ or $o(n^{-3})$, as well as all terms which are odd in any one of the variables v, w_i, u_i because such terms vanish after the integrations in v, w_i, u_i, and using the identity

$$\pi^{-1/2}\int(av^2+bw_i^2)\exp(-v^2-w_\iota^2)\,dv\,dw_\iota=\int(a+b)v^2\exp(-v^2)\,dv,$$

a, b, arbitrary constants, we are left with (see the appendix)

$$\pi^{-(n+1)/2}\int\left\{G'\left(\sum_{i=1}^n\frac{1}{n}V(x_{i-1})\right)\right.$$
$$\cdot\left.\int_0^{1/n}\left(V\left(x_{j-1}+u_j\sqrt{n}\,t+w_j\sqrt{n}\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}\right)-V_{j-1}\right)dt\right\}$$
$$\cdot\exp(-u_1^2-\cdots-u_n^2-w_j^2)\,du_1\,du_2\cdots du_n\,dw_j$$
$$=\pi^{-n/2}\int\left\{\frac{1}{2}G'\left(\sum_{i=1}^n\frac{1}{n}V(x_{i-1})\right)\frac{1}{2n^2}v^2V''(x_{j-1})\right.$$
$$\left.+\frac{1}{2}G''\left(\sum_{i=1}^n\frac{1}{n}V_{i-1}\right)\frac{v^2}{2n^2}V'_{j-1}\sum_{i=j+1}^n V'_{i-1}\right\}$$
$$\cdot\exp(-u_1^2-\cdots-u_{n-1}^2-v^2)\,du_1\cdots du_{n-1}\,dv+O(n^{-3});$$

grouping all such terms, we see that the right-hand side of (13) differs only by terms $O(n^{-2})$ from

$$\int\left\{G\left(\sum_{i=1}^n\frac{1}{n}V(x_{i-1}+v/(2n)^{1/2})\right)\right\}\exp(-u_1^2-\cdots-u_{n-1}^2-v^2)\,du_1\cdots du_{n-1}\,dv$$

and the formula (9) has been established.

The remarkable feature of formula (9) is that it is no more complicated in structure nor does it require more computing effort than the standard "rectangle rule" ([2], [6], [9]), whose accuracy is only $O(n^{-1})$.

Generalizations. One may wish to construct formulas of higher accuracy than (9), e.g., by using the identity

$$\int_C\left(\int_0^1 x^m(t)\,dt\right)^2 dW=2\pi^{-1}\int_{-\infty}^{+\infty}du\int_{-\infty}^{+\infty}du'\int_0^1dt\int_0^t ds \tag{20}$$
$$\cdot(u\sqrt{t})^m(u\sqrt{t}+u'(s-t)^{1/2})^m\cdot\exp(-u^2-u'^2)$$

which generalizes (4). The resulting quadrature formulas are difficult to use, and a more sensible approach to increasing accuracy is the use of a function-space analogue of Richardson extrapolation: if J_n is the n-fold integral approximating a Wiener integral J, and if we know that

$$J_n = J + \text{constant} \cdot n^{-2} + o(n^{-2})$$

then we may evaluate J_n for several values of n and extrapolate in the standard manner. An example will be given below.

The formulas above can be generalized to a slightly wider class of functionals. For example, if

$$F[x] = g(x(1))G\left(\int_0^1 V(x(t))\, dt\right),$$

with

$$\left|\int_{-\infty}^{+\infty} g''(y)e^{-y^2}\, dy\right| < +\infty,$$

then one can verify that

$$\int_C F[x]\, dW = \pi^{-n/2} \int \left\{g\left(x_{n-1} + \frac{v}{\sqrt{n}}\right)G\left(\sum_{i=1}^{n} \frac{1}{n} V(x_{i-1}) + \frac{v}{\sqrt{n}}\right)\right\}$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv.$$

It would be interesting to generalize formula (9) to cases where V is not smooth. I conjecture that (9) remains valid if V is only piecewise smooth, with a finite number of discontinuities; a proof has not yet been given.

Finally, problems may occur in which the order of the integration in [9] is too high for use of Gaussian quadrature. Appeal has to be made to Monte-Carlo quadrature, and it is useful to note that the variance reduction technique described in [4] is particularly well suited for use on integrals of the form (9). This variance reduction technique requires the expansion of the integrand in Hermite polynomials of the u_i, $i = 1, \cdots, n-1$; such Hermite polynomials are identical to the Wiener-Hermite polynomials introduced in [3].

An Example. Consider the integral

$$J = \int_C \left\{\left(\int_0^1 x^2(t)\, dt\right)^2\right\} dW$$

used as an example by Cameron [2]. We have $G(y) = y^2$, $V(y) = y^2$, and

$$J_n = \pi^{-n/2} \int \left\{\sum_{i=1}^{n} \frac{1}{n}\left(\sum_{j=1}^{i-1} \frac{u_j}{\sqrt{n}} + \frac{v}{(2n)^{1/2}}\right)^2\right\}^2$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv.$$

The expression in curly brackets squared is a polynomial of degree 4 in u_i, $i = 1, \cdots n-1$, and v; the integral can therefore be evaluated exactly by a finite sum containing 3^n terms, obtained by application of weighted Gaussian quadrature. For example,

$$J_1 = \pi^{-1/2} \int \left(\frac{v}{\sqrt{2}}\right)^4 e^{-v^2}\, dv,$$

$$J_2 = \pi^{-1} \int \left\{\frac{1}{2}\left(\frac{v}{\sqrt{4}}\right)^2 + \frac{1}{2}\left(\frac{u_1}{\sqrt{2}} + \frac{v}{\sqrt{4}}\right)^2\right\}^2 \exp(-u_1^2 - v^2)\, du_1\, dv.$$

It can be shown, (e.g., by application of formula (20)) that $J = 7/48$ (see also [2]). Some tedious but elementary algebra shows that $J_n = J + (1/24)n^{-2}$, so that

$$J_1 = \frac{9}{48}, \qquad J_2 = \frac{15/2}{48}, \qquad J_3 = \frac{65/9}{48},$$

etc. Extrapolation from any two of these values, e.g., $\frac{1}{3}(4J_2 - J_1)$, yields the exact value $J = 7/48$.

Less elementary examples will be displayed in [5].

Appendix. In this appendix, we reproduce some of the intermediate algebraic steps omitted in the main text, in particular those following the change of variable defined by Eq. (19).

Introduce the notation

$$G(\cdot) = G\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}\right).$$

We start at Eq. (19), in which the change of variables

$$v = u_j \tag{19}$$

is made. V_i, for $i \leqq j$, does not depend on u_j; thus G' becomes

$$\begin{aligned} G'\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}\right) &= G'\left(\sum_{i=1}^{j} \frac{1}{n} V_{i-1} + \sum_{i=j+1}^{n} V\left(x_{i-1} + \frac{v - u_j}{\sqrt{n}}\right)\right) \\ &= G'(\cdot) + G''(\cdot)\left(\sum_{i=j+1}^{n} \frac{1}{n} V'(x_{i-1}) \frac{v - u_j}{\sqrt{n}}\right) + \text{terms of order } n^{-2}. \end{aligned}$$

Furthermore, expanding $G''(\cdot)$ in powers of u_j, we obtain

$$\begin{aligned} G''(\cdot) = {} & G''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right) + G'''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right) \frac{1}{n} \sum_{i=j+1}^{n} V_{i-1}^{(j)} \frac{u_j}{\sqrt{n}} \\ & + \text{terms of order } n^{-3}. \end{aligned}$$

Thus G' becomes, after the change (19),

$$\begin{aligned} & G'(\cdot) + G''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right)\left(\sum_{i=j+1}^{n} \frac{1}{n} V'(x_{i-1}) \frac{v - u_j}{\sqrt{n}}\right) \\ & \quad + G'''\left(\sum_{i=1}^{n} \frac{1}{n} V_{i-1}^{(j)}\right) \frac{1}{n}\left(\sum_{i=j+1}^{n} V_{i-1}^{(j)} \frac{u_j}{\sqrt{n}}\right)\left(\sum_{i=j+1}^{n} \frac{1}{n} V'(x_{i-1}) \frac{v - u_j}{\sqrt{n}}\right) \\ & \quad + \text{terms of order } n^{-3}. \end{aligned}$$

We now multiply this expression by

$$\begin{aligned} & \int_0^{1/n} \left\{ V\left(x_{j-1} + v\sqrt{n}\, t + w_j \sqrt{n} \left(t\left(\frac{1}{n} - t\right)\right)^{1/2}\right) - V(x_{j-1}) \right\} dt \\ & \quad = V'_{j-1}\left(v\sqrt{n} \frac{1}{2n^2} + w_j \sqrt{n} \int_0^{1/n} \left(t\left(\frac{1}{n} - t\right)\right)^{1/2} dt\right) \\ & \qquad + \frac{1}{2} V''_{j-1}\left(v^2 \frac{1}{3n^2} + w_j^2 \frac{1}{6n^2} + 2u_j w_j n \int_0^{1/n} t\left(t\left(\frac{1}{n} - t\right)\right)^{1/2} dt\right) + O(n^{-5/2}). \end{aligned}$$

It is important to note that the functions $V_{i-1}^{(i)}$, V_{j-1}, V'_{j-1}, V''_{j-1} do not depend on v or w_j, and thus, we shall obtain polynomials in v and w_j. Terms which include odd powers of either v or w_j can be omitted because, after integration with respect to either v or w_j, they will vanish.

The coefficient of v^2 is

$$\sqrt{n}\,\frac{1}{2n^2}\left(G''\left(\sum_{i=1}^{n}\frac{1}{n}\,V_{i-1}^{(i)}\right)\sum_{i=j+1}^{n}\frac{1}{n}\,V'(x_{i-1})\,\frac{1}{\sqrt{n}}\cdot V'_{j-1}\right)$$
$$+\sqrt{n}\,\frac{1}{2n^2}\,G'''\left(\sum_{i=1}^{n}\frac{1}{n}\,V_{i-1}^{(i)}\right)\frac{1}{n}\left(\sum_{i=j+1}^{n}V_{i-1}^{(i)}\,\frac{u_j}{\sqrt{n}}\right)\sum_{i=j+1}^{n}\frac{1}{n}\,V'_{i-1}\,\frac{1}{\sqrt{n}}$$
$$+\frac{1}{2}\,V''_{j-1}\,\frac{1}{3n^2}\,G'(\cdot)+O(n^{-3}).$$

The term on the second line is a product of u_j and of a function independent of u_j, and will thus vanish after integration with respect to u_j.

The coefficient of w_j^2 is $(1/12n^2)V''_{j-1}G'(\cdot)$. There are no terms in u_j^2 which do not include as a factor either v or w_j. Thus, we find

$$\int\left\{G'(\cdot)\int_0^{1/n}\left(V\left(x_{j-1}+u_j\sqrt{n}\,t+w_j\sqrt{n}\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}\right)-V(x_{j-1})\right)dt\right\}$$
$$\cdot\exp(-u_1^2-\cdots-u_n^2-w_j^2)\,du_1\cdots du_n\,dw_j$$
$$=\int\left\{G'(u_j=v)\int_0^{1/n}V\left(u_{j-1}+v\sqrt{n}\,t+w_j\sqrt{n}\left(t\left(\frac{1}{n}-t\right)\right)^{1/2}-V(x_{j-1})\right)dt\right\}$$
$$\cdot\exp(-u_1^2-\cdots-u_{j-1}^2-v^2-u_{j+1}^2-\cdots-u_n^2-w_j^2)$$
$$\cdot du_1\cdots du_{j-1}\,dv\,du_{j+1}\cdots du_n\,dw_j$$
$$=\int\{G'(\cdot)\exp(-u_1^2-\cdots-u_{j-1}^2-u_{j+1}^2-\cdots-u_n^2)$$
$$\cdot du_1\cdots du_{j-1}\,du_{j+1}\cdots du_n\}$$
$$\cdot\left\{\int\left(\frac{1}{12n^2}\,V''_{j-1}w_j^2+\frac{1}{6n^2}\,V''_{j-1}\,v^2\right)\exp(-v^2-w_j^2)\,dv\,dw_j\right\}$$
$$+\int G''(\cdot)\,\frac{v^2}{2n^3}\,V'_{j-1}\sum_{i=j+1}^{n}V'_{i-1}\exp(-u_1^2\cdots u_{n-1}^2-v^2)\,du_1\cdots du_{n-1}\,dv$$
$$+O(n^{-3}).$$

But we have

$$\int\left(\frac{1}{12n^2}\,w_j^2+\frac{1}{6n^2}\,v^2\right)\exp(-v^2-w_j^2)\,dv\,dw_j$$
$$=\frac{1}{2n^2}\int\frac{1}{2}\,v^2\exp(-v^2)\,dv\cdot\int\exp(-u_n^2)\,du_n.$$

Therefore, the integral above reduces to

$$\int \left\{\frac{1}{2} G'(\cdot) \frac{1}{2n^2} v^2 V''_{j-1} + G''(\cdot) \frac{v^2}{2n^3} V'_{j-1} \sum_{i=j+1}^{n} V'_{i-1}\right\}$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv.$$

Summing over all j and adding the integral

$$\int G(\cdot) \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv$$

we obtain from (13)

$$\int \left\{G(\cdot) + G'(\cdot) \sum_{j=1}^{n} \Delta q_j\right\} dW$$
$$= \int \left\{G(\cdot) + \sum_{j=1}^{n} \frac{1}{2} G'(\cdot) \frac{1}{2n^2} v^2 V''_{j-1} + \sum_{j=1}^{n} G''(\cdot) \frac{v^2}{2n^3} V'_{j-1} \sum_{i>j} V'_{i-1}\right\}$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv$$
$$= \int \left\{G(\cdot) + \sum \frac{1}{2} G'(\cdot) \frac{1}{2n^2} v^2 V''_{j-1} + \frac{1}{2} \sum_{i,j} G''(\cdot) \frac{v^2}{2n^3} V'_{j-1} V'_{i-1}\right\}$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv$$
$$+ \int \left\{-\frac{1}{2} \sum_{i=j} G''(\cdot) \frac{v^2}{2n^3} V'_{j-1} V'_{i-1}\right\}$$
$$\cdot \exp(-u_1^2 - \cdots - u_{n-1}^2 - v^2)\, du_1 \cdots du_{n-1}\, dv.$$

The second integral is of order n^{-2}, the first integral is merely the expansion to order n^{-2} of formula (9).

The proof of formula (9) for cases where $g \neq 1$ is a mere repetition of the previous calculation, with the change of variable (19) performed in the argument of g as well as in the argument of G.

Department of Mathematics
University of California
Berkeley, California 94720

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