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Mesh Refinement

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Abstract. We study mesh refinement techniques for first-order hyperbolic equations. A refinement method for use with the leap-frog scheme is defined and its stability established. The remainder of the paper is devoted to a discussion of the effects of nonuniform grids and the circumstances under which they may be used.

I. Introduction. Often, the solutions of partial differential equations are much smoother in some parts of the region in which a solution is sought than they are in others. Furthermore, there is often good a priori knowledge of this behavior, e.g., boundary layer phenomena. Suppose we want to solve these problems using difference methods. It is well known that the complexity of the solution requires a certain net spacing to obtain given accuracy. It is attractive for economical reasons to consider using different mesh intervals in different parts of the region in this situation (see, e.g., [5]).

In this paper, we consider using refinement techniques for time dependent problems whose behavior is essentially hyperbolic. M. Ciment [1] has established stability for several refinement methods used with dissipative difference schemes. We begin in Section II by establishing stability for a refinement method used with the nondissipative leap-frog scheme and include a computational example of this method.

We then discuss certain problems connected with the use of this technique. It has been stated that the use of nonuniform grid intervals invariably causes reflections. We have seen that this need not be a problem. However, there is another phenomenon which can cause trouble and is, in a sense which is made precise later, intrinsic in the technique and unavoidable. Any wave which is poorly represented in a coarser grid [4] will change phase speed when passing through an interface into a finer grid. If this wave later passes from the fine grid back into the coarse grid, a serious interaction can result with that part of the wave which has remained in the coarse net. We begin this discussion in Section III by first considering the related problem of using difference methods for a first-order hyperbolic equation in a quarter space with homogeneous initial data and inhomogeneous boundary data. This discussion establishes a quantitative estimate of the change in signal speed of waves passing through the interface of two grids. We include computational results which illustrate this phenomenon. A two-dimensional computation illustrates the effect of serious interactions of waves which have become out of phase due to the use of a mesh refinement.

II. Stability of a Refinement Procedure. We now examine a refinement procedure for hyperbolic equations which is similar to a procedure used by E. Isaacson [3] to handle discontinuous coefficients of parabolic equations. Consider the Cauchy

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problem for the equation

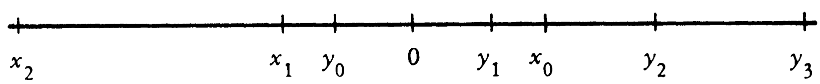
$$w_t = aw_x, \qquad -\infty < x < \infty,\ t \geqq 0, \tag{2.1}$$

with initial values

$$w(x, 0) = f(x). \tag{2.2}$$

We approximate (2.1) using the leap-frog scheme with a refinement at $x = 0$. Let $h_1 > h_2 > 0$ denote the different mesh intervals and define gridpoints by

$$x_\nu = -(\nu - 1/2)h_1, \qquad y_\nu = (\nu - 1/2)h_2, \quad \text{for } \nu = 0, 1, 2, \cdots .$$



Let $k > 0$ denote the time step and

$$u_\nu(t) = u(x_\nu, t), \qquad v_\nu(t) = v(y_\nu, t),$$

$t = 0, k, 2k, \cdots$, the gridfunctions. Approximate (2.1) by

$$u_\nu(t + k) = u_\nu(t - k) - a\lambda_1(u_{\nu+1}(t) - u_{\nu-1}(t)) \tag{2.3}$$

and

$$v_\nu(t + k) = v_\nu(t - k) + a\lambda_2(v_{\nu+1}(t) - v_{\nu-1}(t)) \tag{2.4}$$

for $\nu = 1, 2, \cdots$, where $\lambda_1 = k/h_1$, $\lambda_2 = k/h_2$.

The solution is uniquely determined if we give initial values for $t = 0, k$ and impose the continuity conditions

$$\begin{aligned} u_0 + u_1 &= v_0 + v_1, \\ (u_0 - u_1)/h_1 &= -(v_0 - v_1)/h_2 \end{aligned} \tag{2.5}$$

at the interface $x = 0$.

We assume that (2.3) and (2.4) are stable for the related Cauchy problems, i.e., $0 \leqq a\lambda_1, a\lambda_2 \leqq 1$.

We can consider (2.3), (2.4), (2.5) as an initial-boundary value problem in the quarter-space $x \geqq 0$, $t \geqq 0$. Its stability then follows from the theory in [2]. Associated with (2.3), (2.4), (2.5) is the following resolvent problem:

$$z^2\hat{u}_\nu = \hat{u}_\nu - a\lambda_1 z(\hat{u}_{\nu+1} - \hat{u}_{\nu-1}), \tag{2.6}$$

$$z^2\hat{v}_\nu = \hat{v}_\nu + a\lambda_2 z(\hat{v}_{\nu+1} - \hat{v}_{\nu-1}), \tag{2.7}$$

$$\begin{aligned} \hat{u}_0 + \hat{u}_1 &= \hat{v}_0 + \hat{v}_1 + \hat{g}_1, \\ \hat{u}_0 - \hat{u}_1 &= -p(\hat{v}_0 - \hat{v}_1) + \hat{g}_2, \end{aligned} \tag{2.8}$$

where $p = h_1/h_2 > 1$. The method is stable by Theorem (5.1) of [2] if (2.6)–(2.8) have, for $|z| > 1$, a unique solution with

$$\sum_{\nu=0}^{\infty} (|\hat{u}_\nu|^2 + |\hat{v}_\nu|^2) < \infty \tag{2.9}$$

and there is a constant K such that

$$|\hat{v}_0| + |\hat{u}_0| \leqq K(|\hat{g}_1| + |\hat{g}_2|). \tag{2.10}$$

Let $|z| > 1$; the general solutions of (2.6) and (2.7) satisfying (2.9) are

$$\begin{aligned} \hat{u}_\nu &= \rho_1 \kappa_1^\nu, \qquad \nu = 0, 1, 2, \cdots, \\ \hat{v}_\nu &= \rho_2 \kappa_2^\nu, \qquad \nu = 0, 1, 2, \cdots, \end{aligned} \tag{2.11}$$

where

$$\kappa_j = (-1)^j \left((2\lambda_j a)^{-1} \frac{z^2 - 1}{z} - \left(\left((2\lambda_j a)^{-1} \frac{z^2 - 1}{z} \right)^2 + 1 \right)^{1/2} \right),$$

$|\kappa_j| < 1$, $j = 1, 2$, are the solutions of the characteristic equations

$$\kappa^2 + (-1)^{j+1} (\lambda_j a)^{-1} \frac{z^2 - 1}{z} \kappa - 1 = 0, \qquad j = 1, 2.$$

Substitute (2.11) into (2.8) to obtain

$$C(z) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \equiv \begin{pmatrix} 1 + \kappa_1 & -1 - \kappa_2 \\ 1 - \kappa_1 & p(1 - \kappa_2) \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \hat{g}_1 \\ \hat{g}_2 \end{pmatrix}.$$

It is obvious that the estimate (2.10) holds if

$$\det C(z) = (p + 1)(1 - \kappa_1 \kappa_2) + (p - 1)(\kappa_1 - \kappa_2) \neq 0$$

for $|z| \geqq 1$. Suppose $\det C = 0$, then

$$p \frac{\kappa_1 + 1}{\kappa_1 - 1} = -\frac{\kappa_2 + 1}{\kappa_2 - 1}. \tag{2.12}$$

Since $|\kappa_j| < 1$ for $|z| > 1$, this last relation cannot hold for $|z| > 1$. Let $z = e^{i\theta}$ and $\alpha = (\lambda_1 a)^{-1}$; then

$$\begin{aligned} \kappa_1 &= -i\alpha \sin \theta + (1 - (\alpha \sin \theta)^2)^{1/2}, \\ \kappa_2 &= ip^{-1} \alpha \sin \theta - (1 - (\alpha p^{-1} \sin \theta)^2)^{1/2}. \end{aligned}$$

If $|\alpha \sin \theta| > 1$, then $|\kappa_1| < 1$ and (2.12) cannot hold. If $|\alpha \sin \theta| \leqq 1$, then

$$\begin{aligned} -\operatorname{Im} \det C(z) &= (p + 1)\alpha \sin \theta ((1 - (\alpha p^{-1} \sin \theta)^2)^{1/2} + p^{-1}(1 - (\alpha \sin \theta)^2)^{1/2}) \\ &\quad + \alpha \sin \theta (p - 1)(p^{-1} + 1) \\ &\neq 0 \quad \text{for } \sin \theta \neq 0. \end{aligned}$$

If $\sin \theta = 0$, we have $\kappa_1 = 1$, $\kappa_2 = -1$, so $\det C = 4p \neq 0$. Thus, the approximation is stable.

Consider computing an approximate solution of the problem

$$u_t = -u_x + \epsilon u_{xx}, \qquad 0 \leqq x \leqq 1, \; t \geqq 0,$$

with initial data

$$u(x, 0) = \sin \pi x,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0.$$

We use the scheme

$$v_\nu(t+k) = v_\nu(t-k) - 2kD_0v_\nu(t) + 2\epsilon kD_+D_-v_\nu(t-k),$$

$$v_\nu(0) = \sin \pi\nu h,$$

$$v_0(t) = v_N(t) = 0,$$

where $Nh = 1$, $\lambda = k/h$,

$$D_0v_\nu(t) = (v_{\nu+1}(t) - v_{\nu-1}(t))/2h,$$

$$D_+v_\nu(t) = (v_{\nu+1}(t) - v_\nu(t))/h,$$

and

$$D_-v_\nu(t) = (v_\nu(t) - v_{\nu-1}(t))/h.$$

In Fig. 1, we show the result of this computation for $\epsilon = 10^{-3}$ with $h = 10^{-2}$ and $k = 10^{-3}$ at time $t = 0.52$. We have also computed an approximation using the previously defined refinement procedure. We have a 5:1 refinement in the right half of the interval. The continuity equations (2.5) are centered about the point 0.495. The mesh interval in the coarser part of the net is $h_c = 10^{-2}$ and that in the finer portion $h_f = 0.2 \times 10^{-2}$. For this computation, we have used $k = 10^{-4}$ in both nets. The result at $t = 0.52$ is shown in Fig. 2.

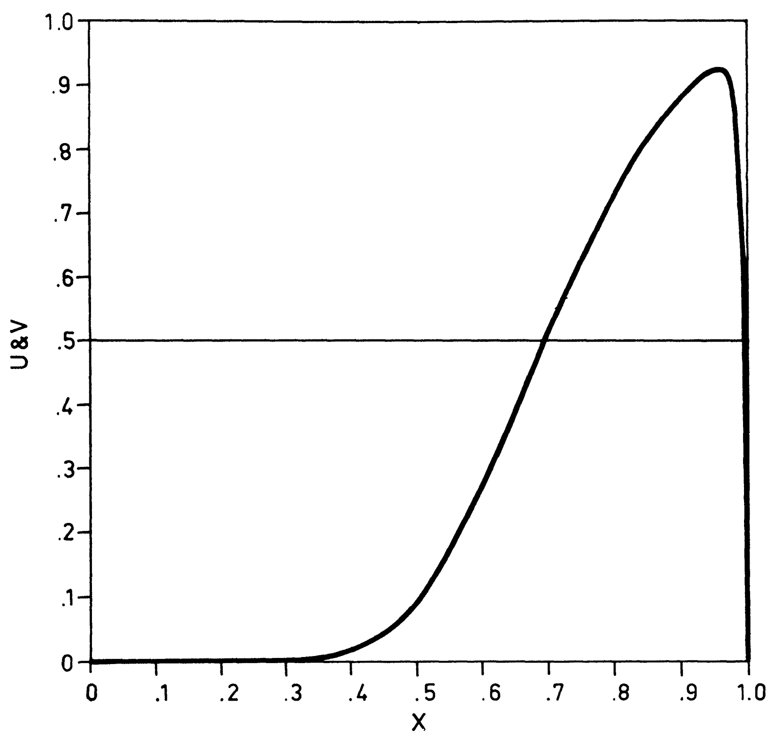


FIGURE 1

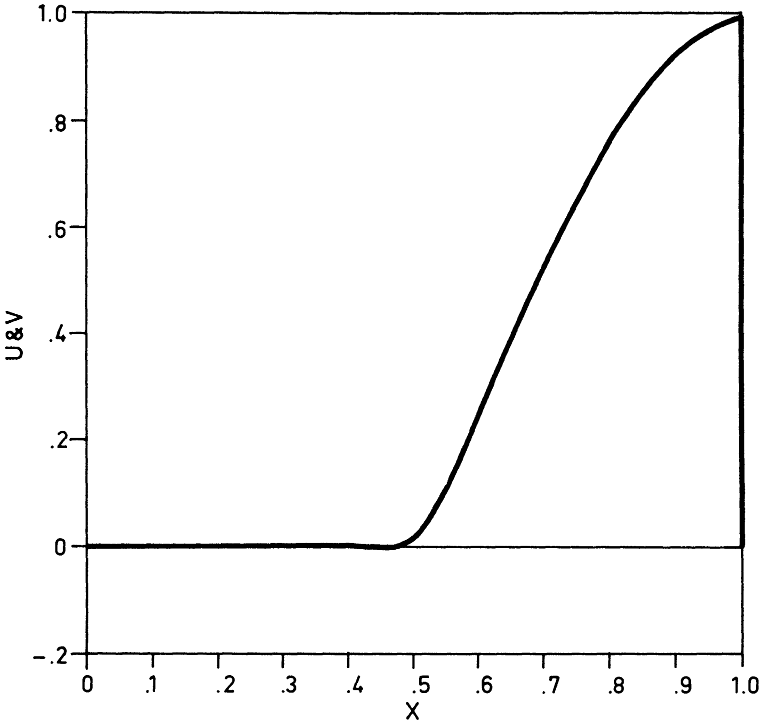


FIGURE 2

The improved accuracy using the refinement is apparent. In practice, the refinement should be introduced much further to the right (nearer 1) to reduce the number of net points, and a larger time step used in the coarser net. This can be done by using the continuity conditions (2.5) at the times nk_c where k_c is the time interval in the coarser net, and then interpolating in the fine net for the first two net points at the intermediate times mk_f. We have used procedures of this type with nearly indistinguishable results.

III. Wave Propagation. Consider the differential equation

$$u_t = -u_x \tag{3.1}$$

for $x \geqq 0$, $t \geqq 0$ with initial values

$$u(x, 0) = 0 \tag{3.2}$$

and boundary condition

$$u(0, t) = e^{i\alpha t}. \tag{3.3}$$

The solution of this problem is given by

$$\begin{aligned} u(x, t) &= e^{i\alpha(t-x)}, \qquad x \leqq t, \\ &= 0, \qquad x > t. \end{aligned} \tag{3.4}$$

Thus, the signal speed is 1.

Approximate this problem by

$$v_\nu(t+k) = v_\nu(t-k) - 2kD_0v_\nu(t), \qquad \nu = 1, 2, \cdots, \tag{3.5}$$

$$v_\nu(0) = v_\nu(k) = 0, \qquad \nu = 1, 2, \cdots, \tag{3.6}$$

$$v_0(t) = e^{i\alpha t}, \qquad t = 0, k, 2k, \cdots. \tag{3.7}$$

This scheme is stable for $\lambda = k/h \leqq 1$ where $k > 0$ denotes a time increment, $h > 0$ the mesh interval in the x-coordinate and

$$v_\nu(t) = v(x_\nu, t), \qquad x_\nu = \nu h,\ t = 0, k, 2k, \cdots.$$

We want to determine the signal speed for the solution to these difference equations. For this reason, we introduce a new variable, $w_\nu(t)$, defined by

$$v_\nu(t) = e^{i\tilde{\alpha}t - \beta x} w_\nu(t), \tag{3.8}$$

where

$$\begin{aligned} \tilde{\alpha} &= \alpha, & \left|\frac{\sin \alpha k}{\lambda}\right| &\leqq 1, \\ &= k^{-1} \arcsin \lambda, & \left|\frac{\sin \alpha k}{\lambda}\right| &> 1. \end{aligned}$$

We substitute (3.8) into (3.5)–(3.7) and obtain

$$\begin{gathered} e^{i\tilde{\alpha}k} w_\nu(t+k) = e^{-i\tilde{\alpha}k} w_\nu(t-k) - kh^{-1}(e^{-\beta h} w_{\nu+1}(t) - e^{\beta h} w_{\nu-1}(t)), \\ w_\nu(0) = w_\nu(k) = 0, \qquad w_0(t) = e^{i(\alpha - \tilde{\alpha})t}. \end{gathered} \tag{3.9}$$

We can rewrite (3.9) as

$$\begin{aligned} e^{i\tilde{\alpha}k}(w_\nu(t+k) - w_\nu(t))k^{-1} &= e^{-i\tilde{\alpha}k}(w_\nu(t-k) - w_\nu(t))k^{-1} \\ &\quad - [e^{-\beta h}(w_{\nu+1}(t) - w_\nu(t))h^{-1} + e^{\beta h}(w_\nu(t) - w_{\nu-1}(t))h^{-1}] \\ &\quad - k^{-1}[(e^{i\tilde{\alpha}k} - e^{-i\tilde{\alpha}k}) - \lambda(e^{-\beta h} - e^{\beta h})]w_\nu(t). \end{aligned} \tag{3.10}$$

If $|\sin(\alpha k)/\lambda| \leqq 1$, then $\tilde{\alpha} = \alpha$ and we can choose $\beta = i\delta$, δ real, such that

$$2i \sin \alpha k = e^{i\alpha k} - e^{-i\alpha k} = \lambda(e^{\beta h} - e^{-\beta h}) = 2i\lambda \sin \delta h. \tag{3.11}$$

We can then consider (3.10) as an approximation to the differential equation

$$y_t = -\frac{\cos \delta h}{\cos \alpha k} y_x \equiv -by_x, \qquad y(x, 0) = 0,\ y(0, t) = 1 \tag{3.12}$$

and the solution of (3.5)–(3.7) is approximately

$$\begin{aligned} v(x, t) &= e^{i(\alpha t - \delta x)} y, & x &\leqq bt, \\ &= 0, & x &> bt. \end{aligned} \tag{3.13}$$

Thus, the signal speed of the solution of the difference approximation is approximately

$$b = \cos \delta h / \cos \alpha k. \tag{3.14}$$

The relation (3.11) can be written

(3.15) $$\sin(\alpha k)/\lambda = \sin \delta h.$$

Let us assume for simplicity that we compute with very small time steps. Then (3.14) and (3.15) become

$$b \approx \cos \delta h \approx (1 - (\alpha h)^2)^{1/2} = (1 - (2\pi/N)^2)^{1/2}, \qquad \sin \delta h \approx \alpha h,$$

where $N = 2\pi/\alpha h$ denotes the number of mesh points per wavelength in the x-direction. This shows that N has to be quite large for b to be near 1. For example:

(3.16)

N	32	16	8	7	6
b	0.98	0.92	0.64	0.48	0

Now assume that $|\sin(\alpha k)/\lambda| > 1$. Then, we have $\tilde{\alpha} = k^{-1} \arcsin \lambda$, $\sin \tilde{\alpha} k = \lambda$ and $e^{i\tilde{\alpha} k} - e^{-i\tilde{\alpha} k} - \lambda(e^{-\beta h} - e^{\beta h}) = 0$ for $\beta h = i\pi/2$. Then (3.10) approximates the differential equation $y_t = 0$ and the forced wave does not propagate into the interior of the region.

If a wave is already well represented [4] in the coarse net, the previous analysis indicates that this wave should propagate through the interface of the coarse and fine nets without difficulty. Our computations confirm this. On the other hand, this analysis also indicates that there will be difficulty with the propagation of any wave which is not well represented in the coarse net through the interface of the nets. We have done several computations which confirm this.

Consider the problem

(3.17) $$u_t = u_x, \qquad x \leqq 1, t \geqq 0,$$

with initial values $u(x, 0) = 0$ and boundary values $u(1, t) = \sin 2\pi\omega t$. We approximate (3.17) by the difference equation

(3.18) $$\begin{gathered} v_\nu(t + k) = v_\nu(t - k) + 2k D_0 v_\nu(t) + 2\epsilon k h^2 D_+ D_- v_\nu(t - k), \\ v_\nu(0) = v_\nu(k) = 0, \qquad v_0(t) = \sin 2\pi\omega t. \end{gathered}$$

We have a mesh refinement to the right of 0.495 as in the previous computational example and $h_c = 10^{-2}$, $h_f = 0.2 \times 10^{-2}$, $k = 10^{-3}$. We use $\omega_i = 25/2, 33/2, 50/2$ for $i = 1, 2, 3$, respectively. In the fine net, we have $N_f(\omega_i) = 500/\omega_i = 40, 30, 20$ mesh points per wavelength and therefore expect a good approximation there. However, in the coarse net, we have only $N_c(\omega_i) = 100/\omega_i = 8, 6, 4$ mesh points per wavelength and the previous analysis indicates that the forced wave should only propagate with the speeds $d(\omega_i) = 0.64, 0, 0$, respectively. Figs. 3, 4, and 5 are the results of these computations at $t = 1.84$. In Fig. 3, we have $\omega = \omega_1 = 25/2$ and $\epsilon = 10^{-6}$, in Fig. 4, $\omega = \omega_2 = 33/2$ and $\epsilon = 0.5 \times 10^{-5}$ and in Fig. 5, $\omega = \omega_3 = 50/2$ and $\epsilon = 10^{-5}$.

We now consider mesh refinement for the two-dimensional problem

(3.19) $$u_t = -u_x - u_y, \qquad (x, y) \in [0, 1] \times [0, 1], t \geqq 0$$

with initial values $u(x, y, 0) = \sin(2\pi(6x + 3y))$ and boundary conditions $u(x, 0, t) = u(x, 1, t)$, $u(0, y, t) = u(1, y, t)$. Define a grid function $v_{\nu,\mu}(t) = v(\nu h, \mu h, t)$ for positive increments $h = \Delta x = \Delta y > 0$, $Nh = 1$, and $t = 0, k, 2k, \cdots, k > 0$. We approximate (3.19) by the leap-frog scheme

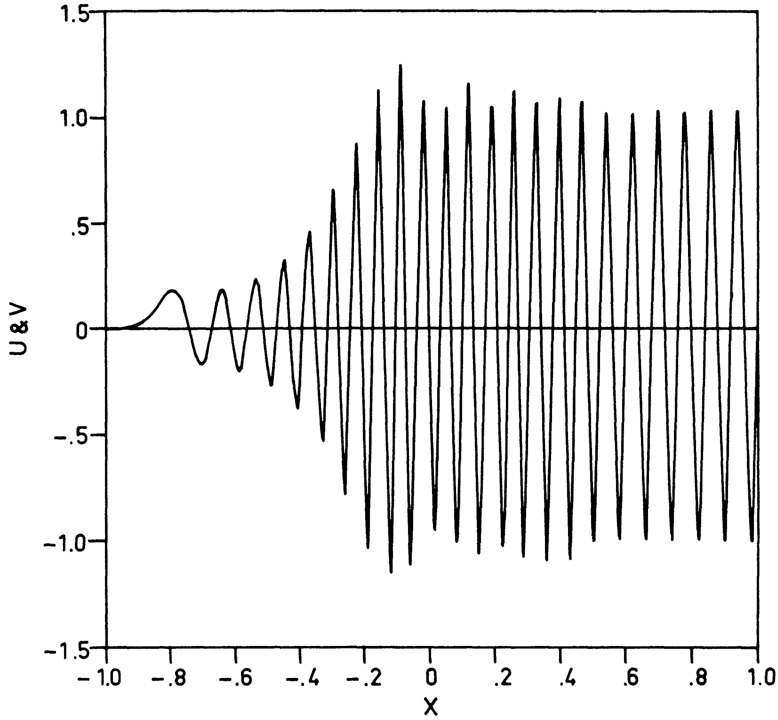


FIGURE 3

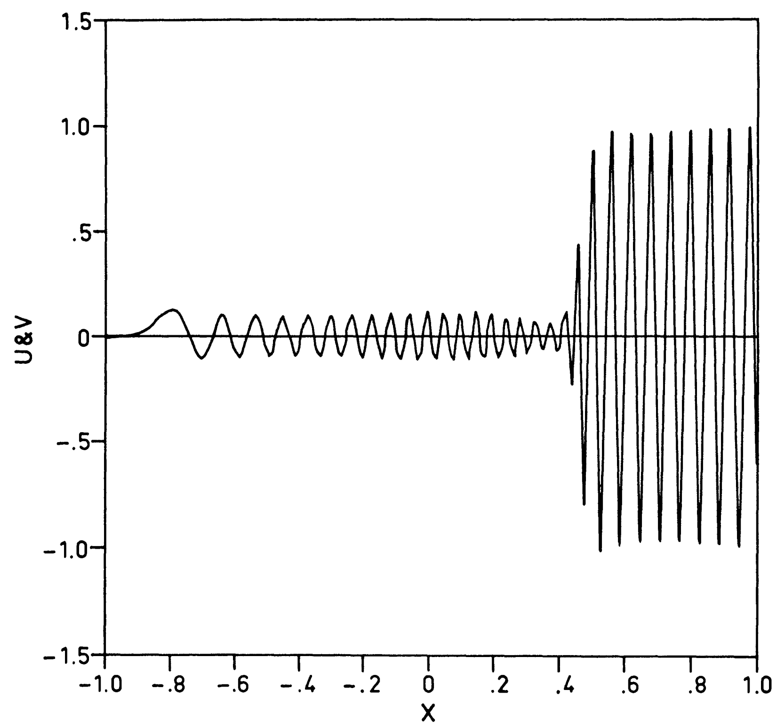


FIGURE 4

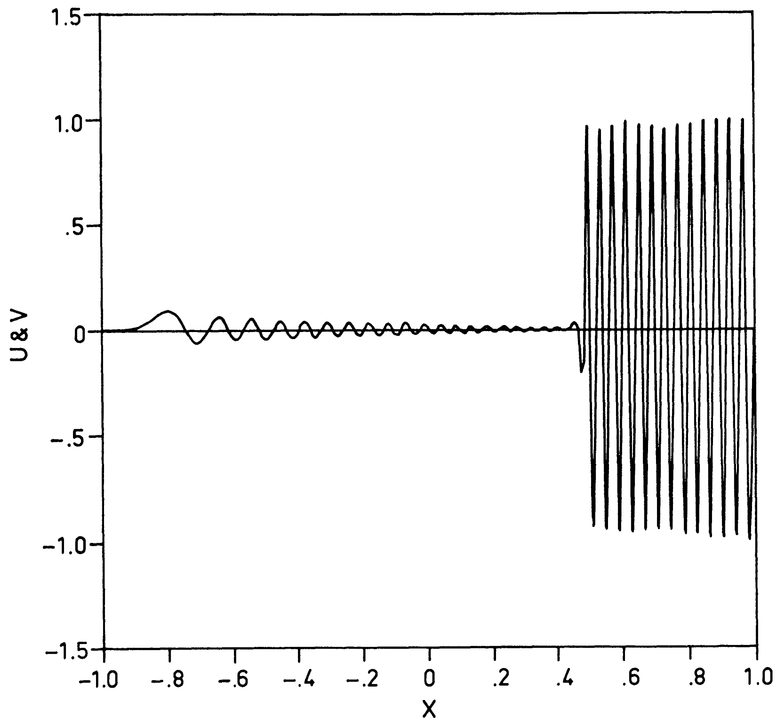


FIGURE 5

$$
\begin{aligned}
v_{\nu,\mu}(t+k) &= v_{\nu,\mu}(t-k) - 2k(D_{0,x} + D_{0,y})v_{\nu,\mu}(t), \\
v_{\nu,\mu}(0) &= v_{\nu,\mu}(k) = \sin(2\pi(6\nu h + 3\mu h)), \qquad (3.20)\\
v_{0,\mu}(t) &= v_{N,\mu}(t), \qquad v_{\nu,0}(t) = v_{\nu,N}(t),
\end{aligned}
$$

where the additional subscripts are used to indicate the coordinate direction in which the previously defined operator acts. We use a 5:1 mesh refinement in the center of the region. We refine about the line segments

$$
\begin{aligned}
&l_1 : x = \alpha, && \alpha \leqq y \leqq \beta, \\
&l_2 : x = \beta, && \alpha \leqq y \leqq \beta, \\
&l_3 : \alpha \leqq x \leqq \beta, && y = \alpha, \\
&l_4 : \alpha \leqq x \leqq \beta, && y = \beta,
\end{aligned}
$$

where $\alpha = 1/3 + 1/90$ and $\beta = 1/3 - 1/90$. We use $h_c = 1/45$ in the coarse net and $h_f = 1/(5 \times 45)$ in the fine net. The refinement procedure is carried out in the same manner as is described in Section II. Linear interpolation is used to provide the additional fine net values required. In the corners of the refined area where the Eqs. (2.5) in the x- and y-directions both define values for the points $(h_f^2 + h_f^2)^{1/2}$ from the corners, we use their average value. This integration was carried out with $k = 10^{-3}$ and the result at $t = 0.75$ is shown in Fig. 6.

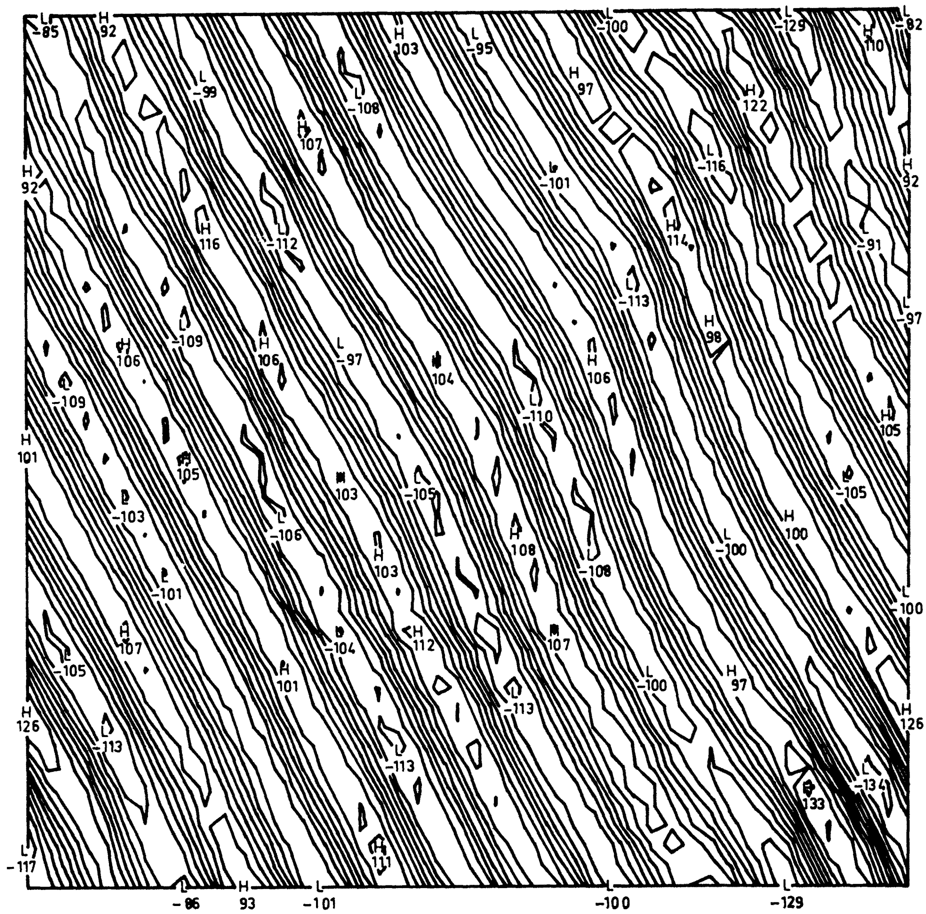


FIGURE 6

The situation here is much worse, as is evidenced by the computation. The dependence of the approximate solution on the mesh intervals produces a phenomenon very much like the propagation of waves in materials of varying density. There is interference of the waves which have passed through the refined region with those which have not. It is obvious how one can construct examples with variable coefficients which double the amplitude at selected points. It is also obvious, since all difference methods have phase errors which are functions of the grid interval, that this phenomenon is present with any difference method used with such a refinement.

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