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A Note on the Catalan-Dickson Conjecture

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Abstract. In this note, the existence of an aliquot sequence with more than 5092 monotonically increasing even terms is proved; the first 125 terms are listed.

The conjecture of Catalan [2], as revised by Dickson [4], is that each aliquot sequence (defined below) is bounded. For the aliquot sequences with leader $n < 276$, the conjecture has been found to be correct. G. A. Paxson [7] calculated $s^{67}(276)$, H. Cohen [3] reached $s^{118}(276)$ and, recently, D. H. Lehmer [6] went up to $s^{362}(276)$, a 32-digit number, $s^{357}(276)$ being a 33-digit number. Guy and Selfridge [5] give more than one hundred sequences which they do not know to be bounded and they believe the Catalan-Dickson conjecture to be false. Further recent work on aliquot sequences can be found in [1]. In this note we construct an aliquot sequence with more than 5092 monotonically increasing terms.

Let n be a natural number, then $\sigma(n)$ is the sum of the divisors of n (including 1 and n), $s(n) = \sigma(n) - n$, the sum of the aliquot divisors of n, and $t(n) = \sigma(n) + s(n)$. The sequence $n = s^0(n), s(n), s^2(n), \cdots, s^i(n), \cdots$, where the exponent denotes iteration, is called the aliquot sequence with leader n. Likewise, we consider the sequence $n = t^0(n), t(n), t^2(n), \cdots, t^i(n), \cdots$. Let p_i denote the ith prime ($p_1 = 2$, $p_2 = 3, p_3 = 5, \cdots$). If the prime factorization of n is given by $n = q_1^{e_1} q_2^{e_2} \cdots q_s^{e_s}$, with $q_1 < \cdots < q_s$ and $e_i \geqq 1$ ($i = 1, \cdots, s$), then

$$(1) \qquad \sigma(n) = \prod_{i=1}^{s} (1 + q_i + \cdots + q_i^{e_i}).$$

The σ-function is multiplicative: $\sigma(uv) = \sigma(u)\sigma(v)$ if $(u, v) = 1$. An upper bound for $t(m)$, without requiring the knowledge of the factorization of m, is given by the following

LEMMA. *If m is odd and $m < \prod_{i=2}^{k+1} p_i$, then $t(m)/m < \prod_{i=1}^{k} p_i/(p_i - 1) - 1$.*

Proof. Let the prime factorization of m be given by $m = \prod_{i=1}^{s} q_i^{e_i}$, $q_1, \cdots, q_s$ being odd primes with $q_1 < \cdots < q_s$, then with (1)

$$\frac{\sigma(m)}{m} = \prod_{i=1}^{s} (1 + q_i^{-1} + \cdots + q_i^{-e_i}) < \prod_{i=1}^{s} (1 + q_i^{-1} + \cdots) = \prod_{i=1}^{s} \frac{q_i}{q_i - 1}.$$

Because $p_2 \leqq q_1, p_3 \leqq q_2, \cdots, p_{s+1} \leqq q_s$, and $s \leqq k - 1$, we conclude

$$\frac{\sigma(m)}{m} < \prod_{i=2}^{s+1} \frac{p_i}{p_i - 1} \leqq \prod_{i=2}^{k} \frac{p_i}{p_i - 1}.$$

Now $t(m) = 2\sigma(m) - m$; therefore,

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$$\frac{t(m)}{m} = 2\frac{\sigma(m)}{m} - 1 < 2\prod_{i=2}^{k}\frac{p_i}{p_i - 1} - 1,$$

which proves the lemma.

THEOREM 1. *Let v denote an even perfect number, $v = 2^{p-1}q$, $q = 2^p - 1$, q prime, $\sigma(v) = 2v$, and let m be an odd number such that $(q, m) = 1$; then the terms of the aliquot sequence $n_0 = vm$, $n_1 = s(n_0), \cdots, n_{i+1} = s(n_i), \cdots$ increase monotonically as long as $(q, t^i(m)) = 1$.*

TABLE 1. $t^i(27)$ $(i = 0, 1, \cdots, 125)$

i	$t^i(27)$	*factorization*
0	27	3(3)
1	53	53
2	55	5·11
3	89	89
4	91	7·13
5	133	7·19
6	187	11·17
7	245	5·7(2)
8	439	439
9	441	3(2)7(2)
10	1041	3·347
11	1743	3·7·83
12	3633	3·7·173
13	7503	3·41·61
14	13329	3(2)1481
15	25203	3·31·271
16	44429	7·11·577
17	66547	13·5119
18	76813	11·6983
19	90803	90803
20	90805	5·11·13·127
21	167243	29·73·79
22	187957	7·11·2441
23	280907	11·25537
24	332005	5·23·2887
25	499739	499739
26	499741	11·181·251
27	600995	5·120199
28	841405	5·168281
29	1177979	11·107089
30	1392181	7·17·11699
31	1977419	269·7351
32	1992661	11·107·1693
33	2398187	11·23·9479
34	3062293	13·103·2287
35	3600363	3·13·92317
36	6739253	23·79·3709
37	7507147	19·395113
38	8297413	8297413
39	8297415	3(2)5·7(2)53·71
40	26274681	3(2)97·30097
41	50415023	191·263953
42	50943313	19·137·19571
43	57094127	29·167·11789
44	61749073	883·69931
45	61890703	7·8841529
46	79573777	79·251·4013
47	82270703	19·4330037
48	90930817	79·379·3037
49	93779583	3·211·148151
50	157486209	3·463·113381
51	263387775	3·5(2)7·501691
52	731969153	17·257·167537
53	824123791	73·167·67601
54	856732337	31·27636527
55	912005455	5·241·863·877
56	1290945713	1290945713
57	1290945715	5·79·479·6823
58	1853553485	5·7·43·1231597
59	3348716467	113·29634659
60	3407986013	71·47999803
61	3503985763	43·7309·11149
62	3668586237	3(2)43·457·20743
63	7200274051	73877·97463
64	7200616733	1103·1733·3767
65	7225819363	7·1531·674239
66	9301151517	3·7·442911977
67	19045215075	3·5(2)809·313889
68	44009008125	3·5(4)47·73·6841
69	107834786307	3(2)77167·155269
70	203693973053	203693973053
71	203693973055	5·17·37·64767559
72	327918159425	5(2)1303·10066559
73	485943083455	5·97188616691
74	680320316849	68032031684
75	680320316851	7(3)1983441157
76	906432609549	3·29·7793·1336939
77	1594393876851	3·531464625617
78	2657323128093	3·89·2087·4768817
79	4511927100387	3·7·69239·3103073
80	9238910900253	3·37607·81889993
81	15398840254563	3(2)1801·950017907
82	29111398771053	3·19·27529·18552301
83	52607781078547	353·149030541299
84	52905842161853	73·6047·119850763
85	54373056097603	2549·21331132247
86	54415718367197	11·2089·2368062943
87	64366318903843	11·163979·35684347
88	76070146337117	61·277·4501991261
89	79122496446547	701·245029·460643

TABLE 1 (continued)

i	$t^i(27)$	factorization
90	79349228998733	1259·63025598887
91	79475280199027	163·41777·11670977
92	80454278794925	5(2)7·1429·8467·37997
93	147768664358995	5·29553732871799
94	206876130102605	5·619511·66786911
95	289627391020723	7·41375341574389
96	372378074169517	13·3121·9491·967019
97	430009354707923	19·1013·22341630109
98	476167162553677	7(2)37·61·97·211(2)997
99	698905653160211	7(2)29·37·41·307·1056089
100	1076553314830189	7(2)149027·147426143
101	1428097751405459	163·293·29902170301
102	1455428335157005	5·11(3)419·521949809
103	2395809654916595	5·479161930983319
104	3354133516883245	5·19·1506257·23440003
105	5119472933924435	5·23·71·101·6207942539
106	8010772510038445	5·7·444803·514563109
107	13961681529683795	5·733·87679·43447837
108	19592451157194925	5(2)3559903·220145899
109	28996841582408275	5(2)17·3329·127261·161047
110	47169396879369005	5·17·554934080933753
111	72696364602321859	7·2040559·5089387243
112	93466835951544381	3·31155611983848127
113	155778059919240643	167·3245101·287449529
114	157643763100651517	53231·2961502941907
115	157649686106641795	5·11·37·139·557331893683
116	269311131006796925	5(2)10772445240271877
117	398580473890059511	1217·3271·100125545873
118	399479401048877897	11·31·17683·20147·3288317
119	500323684776998071	11·1249·10589·20411·168491
120	592328805483801929	83·47387·150600199049
121	606627089589873271	73·341461·24336483907
122	623250611702764137	3(2)5986559·11567591327
123	1177251456351579543	3(2)11·2503·76147·62390777
124	2534405419238744169	3(3)811·4657·57179·434659
125	4985966638665639831	

Proof. Let $m_j = t^i(m)$; then $m_{j+1} = t(m_j) = 2s(m_j) + m_j > m_j$ and, if m_j is odd then also m_{j+1} is odd. Since $m = m_0$ is odd, m_j is odd ($j = 0, 1, 2, \cdots$). If for $k = 0, 1, \cdots, i$, the conditions $n_k = vm_k$ and $(q, m_k) = 1$ are satisfied, then

$$n_{i+1} = s(n_i) = \sigma(vm_i) - vm_i = vt(m_i) = vm_{i+1} > n_i;$$

if, moreover, $(q, m_{i+1}) = 1$, the conditions are satisfied for $k = i + 1$. The conditions are satisfied for $i = 0$ and, hence, $n_0 < n_1 < \cdots < n_i < n_{i+1}$ as long as $(q, m_i) = 1$.

As long as $t^i(m) < q$, obviously $(q, t^i(m)) = 1$. Therefore, we take for v the largest even perfect number known [8]. This gives

THEOREM 2. *Let $v = 2^{19936}q$, $q = 2^{19937} - 1$, q prime; then the terms of the aliquot sequence $v \cdot 27$, $s(v \cdot 27)$, $\cdots$, $s^i(v \cdot 27)$, $\cdots$ increase monotonically at least up to $s^{5092}(v \cdot 27)$.*

Proof. According to Theorem 1 the terms of the sequence are given by $v \cdot 27$, $s^{i+1}(v \cdot 27) = v \cdot t^{i+1}(27)$ ($i = 0, 1, 2, \cdots$), as long as $(q, t^i(27)) = 1$. Now $\prod_{j=2}^{1647} p_j < q < \prod_{j=2}^{1648} p_j$; for the terms $t^i(27) < q$ we find with the lemma:

$$t^{i+1}(27) < \left(\prod_{j=1}^{1647} \frac{p_j}{p_j - 1} - 1\right) t^i(27) < 16.0132\, t^i(27).$$

Table 1 gives the first 125 terms of the sequence $t(27)$, $t^2(27)$, $\cdots$. In order to apply Theorem 1, the condition

$$t^{125+i}(27) < t^{125}(27)(16.0132)^i < q$$

has to be satisfied; this is so, if $i \leqq 4967$. Therefore, the number of monotonically increasing terms of the sequence $t^i(27)$ $(i = 1, 2, \cdots)$, and thus of the aliquot sequence $s^i(v \cdot 27)$ $(i = 1, 2, \cdots)$, is certainly $125 + 4967 = 5092$.

The behaviour of the first 125 terms of the sequence, shown in Table 1, suggests that the actual number of monotonically increasing terms $t^i(27) < q$ is much greater than 5092. From the term $t^j(27)$ that equals or exceeds q onwards, it is possible that a term contains the prime factor q; from that term the behaviour of the terms of the aliquot sequence becomes virtually unpredictable.

Finally, we remark that if there are an infinity of even perfect numbers, it is possible to construct, given any number $N > 0$, an aliquot sequence whose first N terms are monotonically increasing.

Remark. Mr. H. W. Lenstra, Jr. (University of Amsterdam) has communicated to the author a proof of the fact that it is possible to construct such a sequence.

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