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A Probabilistic Approach to a Differential-Difference Equation Arising in Analytic Number Theory

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Abstract. The differential-difference equation

$$\begin{aligned} tv'(t) + v(t-1) &= 0, & t &> 1, \\ v(t) &= 0, & t &< 0, \\ v(t) &= \text{constant}, & 0 &\leqq t \leqq 1, \end{aligned}$$

can be solved by the Monte-Carlo method, for the initial condition $v(t) = e^{-\gamma}$, $0 \leqq t \leqq 1$, where the $v(t)$ represent the probability density of a random variable:

$$t = \lim_{n\to\infty} \sum_{i=1}^{n} \prod_{j=1}^{i} x_j,$$

where the x_j are independent and uniformly distributed on (0, 1).

I. Introduction. The function $\psi(x, y)$ is equal to the number of integers less than or equal to x and free of prime factors greater than y. Chowla and Vijayaraghavan, Ramaswami, Buchstab and de Bruijn have shown that [1]:

$$\lim_{y\to\infty} \frac{\psi(y^t, y)}{y^t} = v(t),$$

where $v(t)$ is a function satisfying

$$\begin{aligned} tv'(t) + v(t-1) &= 0, & t &> 1, \\ v(t) &= 0, & t &< 0, \\ v(t) &= 1, & 0 &\leqq t \leqq 1. \end{aligned}$$

Many authors have studied the limits and asymptotic behaviour of this equation [2]; Norton gives an exhaustive bibliography [3]. Highly accurate numerical results were obtained by Dickman, Bellman, Van de Lune ([4], [5], [6]).

The differential-difference equation solution by the Monte-Carlo method does not claim to be as accurate as these previous calculations but only shows a probabilistic aspect of this equation.

II. Stochastic Model. Let u_n be the random variable: $u_n = x_1 + x_1x_2 + \cdots + x_1x_2 \cdots x_n$, where x_i are independent random variables uniformly distributed on (0, 1).

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It may be deduced from the distribution of a product of x_i variables that if $n \to \infty$, u_n converges in probability to a limit.

LEMMA. *Assume that $v(t)$ is a function continuous on $0 < t < \infty$ satisfying the following equation*:

$$
(1) \qquad \begin{aligned} tv'(t) + v(t-1) &= 0, & t &> 1, \\ v(t) &= 0, & t &< 0, \\ v(t) &= C, & 0 &\leqq t \leqq 1. \end{aligned}
$$

This function is identical to $f(t)$: the probability density of a random variable:

$$t = \lim_{n\to\infty} \sum_{i=1}^{n} \prod_{j=1}^{i} x_j,$$

where x_j are independent random variables uniformly distributed on (0, 1) *if the constant C equals $e^{-\gamma}$, γ being the Euler constant.*

*Proof.** Introduce

$$t_a = \sum_{i=1}^{\infty} \prod_{j=1}^{i} x_j \quad \text{and} \quad t_b = \sum_{i=2}^{\infty} \prod_{j=2}^{i} x_j;$$

t_a and t_b have the same probability distribution and $t_a = x_1(1 + t_b)$, t_b and x_1 are independent.

Let $F(t)$ be the distribution function of t_a:

$$F(t) = \mathcal{P}\mathrm{r}[t_a \leqq t];$$

of course, if $t < 0$, then $F(t) = 0$.

If $t > 0$, we have

$$
\begin{aligned}
F(t) &= \mathcal{P}\mathrm{r}[t_a \leqq t] = \mathcal{P}\mathrm{r}[x_1(t_b + 1) \leqq t] \\
&= \sum \mathcal{P}\mathrm{r}[t_b + 1 \leqq t/x]\mathcal{P}\mathrm{r}[x \leqq x_1 \leqq x + dx] \\
&= \sum F\left(\frac{t}{x} - 1\right)\mathcal{P}\mathrm{r}[x \leqq x_1 \leqq x + dx] = \int_0^1 F\left(\frac{t}{x} - 1\right) dx.
\end{aligned}
$$

Put $(t/x) - 1 = s$, then

$$F(t) = t \int_{t-1}^{\infty} \frac{F(s)}{(s+1)^2}\, ds.$$

If $0 \leqq t \leqq 1$, then

$$F(t) = t \int_0^{\infty} \frac{F(s)\, ds}{(s+1)^2} = C \cdot t,$$

where C is a constant. Hence, $f(t) = F'(t) = C$ for $0 \leqq t \leqq 1$.

If $t > 1$, by differentiating once, we get

$$f(t) = (F(t) - F(t-1))/t \geqq 0;$$

by differentiating again, we find $tf'(t) = -f(t-1)$, $t > 1$.

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* I am indebted to J. J. A. M. Brands for the correction of my initial proof.

TABLE I

n	$\mathcal{P}r\ (u_n \leq 1)$ Explicit value	Monte-Carlo value (10^5 - runs)
2	0.69315	0.69416
3	0.61428	0.61622
4	0.58498	0.58350
5	0.57246	0.57356
6	0.56674	0.57016
7	0.56404	0.56303
8	0.56273	0.56290
9	0.56209	0.56381
10	0.56177	0.56030
∞	0.56146	

Let $h(s)$ be the Laplace transform of $f(t)$ [7]:

$$h(s) = (C_0/s)\exp\{-E_1(s)\},$$

where

$$E_1(s) = \int_s^\infty \frac{e^{-z}\,dz}{z}.$$

TABLE II

t	v(t) Explicit value	t $\Delta t = 0.1$	Monte-Carlo value (20 000 runs)	
			Rough value	Smooth value using REINSCH's (10) program
0		0 -0.1	0.96801	
		0.1-0.2	0.99206	
	1	0.2-0.3	1.03391	
		0.3-0.4	0.96890	
		0.4-0.5	1.01788	
		0.5-0.6	1.01432	
		0.6-0.7	1.02233	
		0.7-0.8	0.99206	
		0.8-0.9	1.01343	
1		0.9-1.0	0.95733	
1.1	0.9046898202	1.0-1.1	0.95911	0.9624
1.2	0.8176784432	1.1-1.2	0.91547	0.8874
1.2	0.7376357355	1.2-1.3	0.81484	0.8132
1.4	0.6635277634	1.3-1.4	0.69016	0.7403
1.5	0.5945348919	1.4-1.5	0.58419	0.6693
1.6	0.5299963708	1.5-1.6	0.57974	0.6006
1.7	0.4693717489	1.6-1.7	0.50849	0.5346
1.8	0.4122133351	1.7-1.8	0.43992	0.4719
1.9	0.3581461138	1.8-1.9	0.37492	0.4128
		1.9-2.0	0.31169	0.3578
2.0	0.3068528194	2.0-2.1	0.29031	0.3070
2.1	0.2604057802	2.1-2.2	0.23510	0.2608
2.2	0.2203571379	2.2-2.3	0.17810	0.2193
2.3	0.1857994616	2.3-2.4	0.17098	0.1826
2.4	0.1559912639	2.4-2.5	0.16208	0.1506
2.5	0.1303195618	2.5-2.6	0.11132	0.1231
2.6	0.1082724430	2.6-2.7	0.09172	0.0999
2.7	0.08941856572	2.7-2.8	0.07748	0.0808
2.8	0.07339158076	2.8-2.9	0.05699	0.0653
2.9	0.05987811599	2.9-3.0	0.05076	0.0528
3.0	0.04860838829	3.0-3.1	0.05165	0.0425
3.1	0.03932296954	3.1-3.2	0.04186	0.0333
3.2	0.03170344451	3.2-3.3	0.02583	0.0250
3.3	0.02546472387	3.3-3.4	0.01514	0.0186
3.4	0.02037177906	3.4-3.5	0.01603	0.0145
3.5	0.01622959324	3.5-3.6	0.00980	0.0125
3.6	0.01287543418	3.6-3.7	0.01069	0.0121
3.7	0.01017283782	3.7-3.8	0.01514	0.0120
3.8	0.008006872188	3.8-3.9	0.00801	0.0092
3.9	0.006280373062	3.9-4.0	0.00534	0.0053
4.0	0.004910925648	4.0-4.1	0.00178	0.0018

Assuming that $f(t)$ is a probability $h(0) = \int_0^\infty f(t)\,dt = 1$, the constant C_0 equals $e^{-\gamma}$, where γ is the Euler constant.

Since $f(t) = C$ as $t = 0$, we obtain the boundary condition: $\lim_{s\to\infty} sh(s) = C = e^{-\gamma}$.

From $f(t) = C$ as $t = 1$, inverting Laplace transform, it may be deduced again that $f(1) = e^{-\gamma}$, so that

$$f(t) = 0, \qquad t < 0,$$

$$f(t) = e^{-\gamma}, \qquad 0 \leqq t \leqq 1,$$

$$f'(t) = -f(t-1)/t, \qquad t > 1.$$

TABLE II

t	v(t) explicit value (*)		Δt = 0.1 t	Monte-Carlo value (3.10^5 runs)	
4.1	0.38285853	10^{-2}	4.1-4.2	0.39	10^{-2}
4.2	0.29754751	10^{-2}	4.2-4.3	0.35	10^{-2}
4.3	0.23050507	10^{-2}	4.3-4.4	0.27	10^{-2}
4.4	0.17799428	10^{-2}	4.4-4.5	0.165	10^{-2}
4.5	0.13701182	10^{-2}	4.5-4.6	0.135	10^{-2}
4.6	0.10514453	10^{-2}	4.6-4.7	0.13	10^{-2}
4.7	0.80455901	10^{-3}	4.7-4.8	0.095	10^{-2}
4.8	0.61395778	10^{-3}	4.8-4.9	0.065	10^{-2}
4.9	0.46728046	10^{-3}	4.9-5.0	0.085	10^{-2}
5.0	0.35472534	10^{-3}	5.0-5.1	0.08	10^{-2}
(Δ) 5.1	0.268580	10^{-3}			
5.2	0.202822	10^{-3}			
5.3	0.152768	10^{-3}			
5.4	0.114775	10^{-3}			
5.5	0.860192	10^{-4}			
5.6	0.643153	10^{-4}			
5.7	0.479771	10^{-4}			
5.8	0.357089	10^{-4}			
5.9	0.265188	10^{-4}			
6.0	0.196503	10^{-4}			

(*)Calculated by 4^{th} order TAYLOR's expansion
(Δ)Calculated by 5^{th} order TAYLOR's expansion

DICKMAN result Monte-Carlo value

$$\int_0^\infty \frac{v(t)}{(1+t)^2}\,dt = 0.62433 \qquad = 0.6238$$

III. Numerical Calculations. For $t \leqq 4$, the solution of Eq. (1) is obtained by explicit expression (see Appendix); for $t > 4$, it is impossible to express the solution by means of known functions. This explicit expression can thus be used for the well-known equation of the statistic theory of damage [8].

$$\begin{aligned} tu'(t) &= u(t-1), && t > 1, \\ u(t) &= 0, && t < 0, \\ u(t) &= 1, && 0 \leqq t \leqq 1. \end{aligned}$$

For $t \leqq 4$, the function $v(t)$ can be calculated with an accuracy depending solely on the polylogarithms which are used in its expression [9]. The random variable u_n is very easy to simulate by means of the pseudo-random numbers of Lehmer's method.

It can be seen in Section II that the u_n distributions achieve rapid convergence as n increases.

For the calculations, n is chosen so that we cannot discriminate between the distributions of u_n and u_{n-1} because the statistical fluctuations of the pseudo-random numbers are greater than the discrepancy between them.

IV. Results. Table I gives an illustration of Section II; notice that we get the Euler constant simulated by $-\operatorname{Log}|\mathcal{P}r[u_n \leqq 1]|$, $n \to \infty$.

Table II represents the calculation of the function $v(t)$ explicitly and by simulation. Results are smoothed by the spline method [10]. Polylogarithms can be calculated by means of Chebyshev's polynomial expansion [11], [12]; Kölbig gives an excellent algorithm for the dilogarithm's calculation [13].

V. Conclusion. The main purpose of this paper is to test the ability of the Monte-Carlo method to resolve differential-difference equations, and, using a classical example, to justify further studies in the field of the statistical theory of damage and neutron transport problems [14] which involve the same mathematical data.

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Appendix.

$v(t)$ explicit behaviour,

$$v(t) = 1 - \operatorname{Log} t, \qquad 1 \leqq t \leqq 2,$$

$$v(t) = 1 - \operatorname{Log} t + [\tfrac{1}{2} \operatorname{Log}^2 t + L_2(1/t) + L_2(-1)], \qquad 2 \leqq t \leqq 3,$$

$$\begin{aligned}
v(t) = {} & 1 - \operatorname{Log} t + [\tfrac{1}{2} \operatorname{Log}^2 t + L_2(1/t) + L_2(-1)] \\
& - \Bigg\{ \frac{1}{4}\left[L_3\left(\frac{1}{4}\right) - L_3\left(\frac{1}{(t-1)^2}\right)\right] - \frac{1}{3}(\operatorname{Log}^3(t-1) - \operatorname{Log}^3 2) \\
& + \frac{1}{2}(\operatorname{Log}^2(t-1)\operatorname{Log} t - \operatorname{Log}^2 2 \operatorname{Log} 3) + L_2\left(\frac{1}{t-1}\right)\operatorname{Log}\frac{t}{t-1} \\
& - L_2\left(\frac{1}{2}\right)\operatorname{Log}\left(\frac{3}{2}\right) - L_2\left(-\frac{1}{t-1}\right)\operatorname{Log}(t-2) + L_2(-1)\operatorname{Log}\frac{t}{3} \\
& + \underbrace{\left\{\left[\operatorname{Log}\frac{1}{2} - \operatorname{Log}\frac{t-2}{t-1}\right] + \left[\frac{1}{2} - \frac{1}{t-1}\right]\right\}}_{V_1} \\
& - \frac{1}{2^2}\underbrace{\left[V_1 + \frac{1}{2}\left(\frac{1}{2^2} - \frac{1}{(t-1)^2}\right)\right]}_{V_2} \\
& + \cdots + \frac{(-1)^{p+1}}{(p+1)^2}\left[V_p + \frac{1}{p+1}\left(\frac{1}{2^{p+1}} - \frac{1}{(t-1)^{p+1}}\right)\right] + \cdots \Bigg\}, \qquad 3 \leqq t \leqq 4.
\end{aligned}$$

By means of Newton's method, the explicit expression permits easy calculation of the roots t_k

$$v(t_k) = \frac{1}{k}, \qquad k = 4, 5, \cdots, 203.$$

For example, the roots

$$t_4 = 2.1245966, \qquad t_5 = 2.2571089$$

are used by Davenport and Erdös [15].

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