REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

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1 [2.05].—ARTHUR SARD & SOL WEINTRAUB, *A Book of Splines*, John Wiley & Sons, New York, 1971, x + 817 pp., 24 cm. Price $22.50.

This book consists of three descriptive and expository chapters totalling 93 pages, an appendix of 12 pages describing a Fortran program for spline approximation, a bibliography of 13 pages inclusive through March 1, 1970, and three elaborate tables of numerical data comprising 693 pages.

The point of view taken throughout is that a spline ξ which approximates an unknown function x is based on linear scalar observations $F_0x, \cdots, F_mx$ of x and the knowledge of the existence of $\int_I (D^nx)^2$ over some interval I. The splines computable from the tables are based on the particular observations $F_ix = x(c + ih)$, $i = 0, \cdots, m$, where c and h are real numbers with $h > 0$, $m = n(1)20$, and $n = 2(1)5, 8$. Chapters 1 and 2 are devoted to instruction on the use of the tables. They are self-contained and written independently of each other, though Chapter 2 is more complete. Thus, Chapter 1, through examples, instructs the reader on how to construct the basis of cardinal spline functions if $m \geqq n$ (or, alternatively, the Lagrange interpolation basis functions if $m = n - 1$) and discusses approximate differentiation and integration by the use of these exact operations on the natural spline interpolant. Chapter 2, however, includes in addition a discussion of the method of calculation of error in natural spline interpolation. The reader is shown here how to compute sharp upper bounds for $|x(t) - \xi(t)|$, $t \in I$. More complete use of the tables is required here.

Chapter 3 draws the entire book together by means of a self-contained exposition of the theory of splines. Eight distinct characterizations of ξ, four of them involving optimality, are presented here. In the interests of clarity and completeness, we shall present these now.

Let I be a compact interval of the real line and let X denote the space of functions x on I whose $(n - 1)$th derivative $D^{n-1}x$ is absolutely continuous on I and whose nth derivative D^nx is square-integrable on I. Here $n \geqq 1$. Let X^* denote the set of functionals Ψ of the form

$$\Psi x = \sum_{\nu=0}^{n-1} \int_I D^\nu x(s)\, d\sigma_\nu(s) + \int_I D^n x(s)\psi(s)\, ds, \qquad x \in X,$$

where $\sigma_0, \cdots, \sigma_{n-1}$ are functions of bounded variation on I and ψ is square-integrable. X^* is precisely the space of continuous linear functionals on X when the latter is normed by

$$||x||^2 = \sum_{\nu=0}^{n-1} |D^\nu x(a)|^2 + \int_I (D^n x)^2 \qquad (a \in I).$$

Now let $F_0, \cdots, F_m$, $m \geqq n - 1$, be fixed functionals in X^* which are linearly inpendent and set $q = 2n - 1$. We define the key functions

$$f_i(t) = F_{i,s}(|t - s|^q), \qquad t \in \mathbf{R}, \; i = 0, \cdots, m.$$

If $I_0 \subset I$ is the smallest closed interval which contains the support of $F_0, \cdots, F_m$, let $a \in I_0$ and define matrices P, Φ and A as follows. (Row indices are denoted by i and column indices by ν.)

$$P = F_i[(s - a)^\nu], \qquad i = 0, \cdots, m, \nu = 0, \cdots, n - 1;$$

$$\Phi = F_i(f_\nu), \qquad i = 0, \cdots, m, \nu = 0, \cdots, m;$$

$$A = \begin{bmatrix} 0 & P^* \\ P & \Phi \end{bmatrix}.$$

Here 0 is an $n \times n$ matrix of zeros and * denotes transpose. The matrices Φ and A are symmetric and A is invertible if and only if P is of full rank n. We assume A to be invertible (valid for the point functionals of the tables) and we write $B = A^{-1}$ as

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = B^*,$$

where the partitioning is consistent with that of A. We introduce the notationally convenient column matrices

$$F = (F_i), \qquad i = 0, \cdots, m,$$

$$f = (f_i), \qquad i = 0, \cdots, m,$$

$$\omega = Fx = (F_i x), \qquad i = 0, \cdots, m, x \in X,$$

$$\eta = ((u - a)^i), \qquad i = 0, \cdots, m, u \in \mathbf{R},$$

of functionals, functions, scalars, and functions, respectively. Finally, we define the cardinal spline functions β_i, $i = 0, \cdots, m$, by

$$\beta = (\beta_i) = [B_{2,1} \; B_{2,2}] \begin{bmatrix} \eta \\ f \end{bmatrix}$$

and the spline projector Π on X by

$$\xi = \Pi x = \beta^* \omega.$$

The linear space M of splines is defined to be the range of the linear projector Π and is spanned by $\beta_0, \cdots, \beta_m$. This, then, is the definition of spline as presented by the authors. The remaining seven characterizing properties of ξ and/or M follow.

I. *Characterization of M Via $M^\perp$.* Define on X the Hilbert space inner product

$$(x, y) = \sum_{i=0}^{m} F_i x F_i y + \int_I D^n x D^n y.$$

Let $N = \{x \in X: F_i x = 0, i = 0, \cdots, m\}$. Then $M = N^\perp$ and $N = M^\perp$.

II. *Geometric Property of the Spline Approximation.* Let $G \in X^*$, $d \geqq 0$ and $\omega \in \mathbf{R}^{m+1}$ be prescribed; let

$$\Gamma = \{x \in X : Fx = \omega\} \quad \text{and} \quad \int_I (D^n x)^2 \leqq d^2.$$

Let $\{\xi_0\} = \Pi\Gamma$. Then the set $G\Gamma$ is the closed interval with midpoint $G\xi_0$ and length equal to twice the square root of $J[d^2 - 2(-1)^n q!\omega^* B_{2,2}\omega]$. J is given explicitly by

$$J = \frac{(-1)^n G_t G_s[|t-s|^q] - \gamma^* B\gamma}{2q!}, \qquad q = 2n - 1,$$

and

$$\gamma = \begin{bmatrix} G\eta \\ Gf \end{bmatrix}.$$

III. *Interpolating Property of the Splines.* For each $x \in X$, there is one and only one $\xi \in M$ such that $F\xi = Fx$. Furthermore, $\xi = \Pi x$.

IV. *Minimal Deviation Among Interpolants.* For each $x \in X$, the integral $\int_I (D^n y)^2$ is minimal among all $y \in X$ such that $Fy = Fx$ if and only if $y = \Pi x$.

V. *Minimal Quotient.* For any $G \in X^*$, define the set $\mathcal{H}$ of admissible approximations of G by

$$\mathcal{H} = \{H \in X^* : Hx = Gx \text{ whenever } D^n x = 0\}.$$

Set $R = G - H$ for $H \in \mathcal{H}$. For each such H there is a continuous linear functional Q on $L^2(I)$ (called the quotient of R and D^n) such that $R = QD^n$. Define $H_0 = G\Pi$. Then $H_0 \in \mathcal{H}$ and $||Q||$ is minimal among $H \in \mathcal{H}$ if and only if $H = H_0$, in which case

$$\min ||Q||^2 = J = \sup_{0 \neq \zeta \in N} \left[|G\zeta|^2 \Big/ \int (D^n\zeta)^2 \right].$$

Here N is defined in I and J agrees with the constant of II.

VI. *Minimal Deviation in M.* For each $x \in X$, the integral $\int_I (D^n x - D^n y)^2$ is minimal among $y \in M$ if and only if $D^n y = D^n \xi$ where $\xi = \Pi x$, i.e., if and only if $y - \xi$ is polynomial of degree $n - 1$.

VII. *Analytic Description of M.* The splines are precisely those functions ξ on R which are such that

(i) ξ is a linear combination of the key functions $f_0, \cdots, f_m$ plus a polynomial of degree $n - 1$, and,

(ii) $D^n\xi(u) = 0$ for all $u \in I_0$.

The authors give credit to various authors for their contributions to the characterizations I–VII.

Properties I, II, and III are due to Golomb and Weinberger (in *On Numerical Approximation*, R. E. Langer, Editor, University of Wisconsin Press, Madison, 1959, pp. 117–190), in even greater generality. Property IV, in the case of natural cubic splines, was known to Schoenberg (*Quart. Appl. Math.*, v. 4, 1946, pp. 45–99) though Holladay (*MTAC*, v. 11, 1957, pp. 233–243) appears to have supplied the first proof. Properties IV and VI, in even greater generality, are implicit in Golomb and Weinberger. Walsh, Ahlberg and Nilson (*J. Math. Mech.*, v. 11, 1962, pp. 225–234) explicitly treated a special case of IV and VI. The first explicit treatment, in the generality considered here, appears to be due to DeBoor and Lynch (*J. Math. Mech.*, v. 15, 1966, pp. 953–969).

Property V is due to Sard (*Amer. J. Math.*, v. 71, 1949, pp. 80–91, and *Linear*

Approximation, Amer. Math. Soc., Providence, R.I., 1963) although the precise calculation of J appears to be due to Secrest (*Math. Comp.*, v. 19, 1965, pp. 79–83) in the case of quadrature. Finally, Schoenberg, in a number of interesting papers between 1964 and 1968, discussed the equivalence of III, IV, V, VI, and VII in a number of important special cases.

We shall now return to the discussion of the numerical tables. The first table allows one to obtain the representations of the cardinal spline functions $\beta_i(u)$ in terms of $1, u, \cdots, u^{n-1}, |u|^q, |u-1|^q, \cdots, |u-m|^q$ (alternatively, $1, u, \cdots, u^{n-1}, u_+^q, \cdots, (u-m)_+^q$) when the uniformly spaced knots are chosen to be $0, 1, \cdots, m$. The table also produces the representation of $\beta_i(t)$ in terms of powers and absolutes or pluses when the knots are symmetrically placed about zero at unit spacing. The coefficients in these representations are given accurately to 13S and are based on 30S values obtained on a CDC 6600 computer. The coefficients were computed in three independent ways, viz., by the direct inversion of the matrix A, by the use of the duals of the functionals F_i [Golomb-Weinberger, *loc. cit.*] and by certain recursion formulas due to Greville [privately communicated by Schoenberg]. The authors report that the first of these three methods was the most efficient, despite the fact that A is not a particularly well-conditioned matrix. The authors speculate that the use of the absolute-value functions, rather than the plus functions, in obtaining A provides the symmetry which makes the inversion of A more effective than it otherwise would be. Indeed, A is symmetric about both diagonals if the knots are symmetrically placed.

Now, it is a simple matter, achieved by an elementary affine transformation, to pass to the cardinal spline functions with uniformly spaced knots in general position. In any event, Table 1, with only special exceptions, provides only the *representation* of the desired spline. If the table user wishes to avoid the calculations necessary for evaluation, differentiation, and integration, he may employ the Fortran program provided. The necessary input information for the punched data cards is presented very clearly. The numerical data are taken from Table 1. Sample output data are shown for four illustrative problems.

Table 2 gives the entries of B to 6S for the particular functionals $F_ix = x(i-p)$, $i = 0, \cdots, m = 2p$, where p is half-integral or integral. Table 3 provides the coefficients required to form the Lagrange interpolant when $m = n - 1$ and $n = 2(1)10$. It should be noted that the range of n is different here from that in Tables 1 and 2.

The reviewer was impressed with the clarity and accuracy of the exposition, both in the instructional Chapters 1 and 2 and in the theoretical Chapter 3. In the careful and impartial meting out of credits in Chapter 3, the authors have performed a considerable service. In particular, the fundamental role of the contribution of Golomb and Weinberger is made clear, a fact not sufficiently recognized in the past.

Altogether, this authoritative book should provide a valuable service to both the users of old mathematics and the makers of new mathematics.

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