

## Calculation of the Ramanujan $\tau$ -Dirichlet Series

By Robert Spira

**Abstract.** A method is found for calculating the Ramanujan  $\tau$ -Dirichlet series  $F(s)$ . An inequality connecting points symmetric with the critical line,  $\sigma = 6$ , is proved, and a table is given for  $\Gamma(s)F(s)$  for  $\sigma = 6.0, 6.5$ ,  $t = 0(.25)16$ . Two zeros are found in  $0 < t \leq 16$ ; they appear to be simple and on the critical line.

The Ramanujan  $\tau$ -function (Hardy [1]) is defined by

$$(1) \quad g(x) = x \prod_{k=1}^{\infty} (1 - x^k)^{24} = \sum_{n=1}^{\infty} \tau(n)x^n,$$

and we consider the series

$$(2) \quad F(s) = \sum_{n=1}^{\infty} \tau(n)/n^s.$$

We have

$$(3) \quad \Gamma(s)F(s) = \int_0^{\infty} y^{s-1}g(e^{-y}) dy$$

and

$$(4) \quad \Gamma(12 - s)F(12 - s) = (2\pi)^{2(6-s)}\Gamma(s)F(s).$$

Further,

$$(5) \quad g(e^{-y}) = (2\pi/y)^{12}g(e^{-4\pi^2/y}).$$

To calculate (3), we break the integral at 1. We have

$$(6) \quad \begin{aligned} \int_1^{\infty} y^{s-1}g(e^{-y}) dy &= \int_1^{\infty} y^{s-1} \sum_{n=1}^{\infty} \tau(n)e^{-ny} dy \\ &= \sum_{n=1}^{\infty} \tau(n) \int_1^{\infty} y^{s-1}e^{-ny} dy \\ &= \sum_{n=1}^{\infty} \tau(n)n^{-s} \int_n^{\infty} u^{s-1}e^{-u} du \\ &= \sum_{n=1}^{\infty} \tau(n)n^{-s}\Gamma(s, n) \end{aligned}$$

and, using (5),

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$$\begin{aligned}
 \int_0^1 y^{s-1} g(e^{-y}) dy &= \int_0^1 y^{s-1} (2\pi/y)^{12} \left( \sum_{n=1}^{\infty} \tau(n) e^{-4\pi^2 n/y} \right) dy \\
 (7) \qquad &= (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) \int_1^{\infty} u^{11-s} e^{-4\pi^2 nu} du \\
 &= (2\pi)^{2(s-6)} \sum_{n=1}^{\infty} \tau(n) n^{s-12} \Gamma(12-s, 4\pi^2 n).
 \end{aligned}$$

In order to truncate the series in (6) and (7), we need estimates for  $\tau(n)$  and  $\Gamma(s, y)$ , where  $|\operatorname{Re} s - 6| \leq \frac{1}{2}$ . It has been proved by Rankin that  $\tau(n) = O(n^{29/5})$ , but that, for an infinity of  $n$ ,  $\tau(n) \geq n^{11/2}$ . We content ourselves here with

LEMMA 1. For  $n \geq 1$ ,  $|\tau(n)| \leq 51,000n^8$ .

*Proof.* For  $0 < x < 1$ , using Hardy [1, p. 156],

$$\sum \tau(n)x^n = x \left\{ \prod_{n=1}^{\infty} (1 - x^n)^{24} \right\} = x(1 - 3x + 5x^3 - 7x^6 + \dots)^8$$

and defining  $\epsilon(n)$  by

$$\sum \epsilon(n)x^n = x(1 + 3x + 5x^3 + 7x^6 + \dots)^8,$$

we have  $0 \leq |\tau(n)| \leq \epsilon(n)$ . Thus,

$$|\tau(n)| x^n \leq \sum |\tau(n)| x^n \leq x \left\{ \sum_{j=0}^{\infty} (2j + 1)x^{j(j+1)/2} \right\}^8.$$

Now, write  $x = e^{-y}$  and take  $x = 1 - 1/n$ . Then, for  $n \geq 25$ , we have  $0 < y < .1$ , and

$$\sum_{j=0}^{\infty} (2j + 1)x^{j(j+1)/2} \leq 1 + 2 \sum_{j=1}^{\infty} (j + \frac{1}{2})x^{j^2/2} \leq 1 + 3 \sum_{j=1}^{\infty} je^{-j^2 y/2}.$$

Now,

$$(d/dt)(te^{-yt^2/2}) = e^{-yt^2/2}(1 - yt^2),$$

which is zero for  $t^2 = 1/y < 1$ . Hence,

$$\begin{aligned}
 1 + 3 \sum_{j=1}^{\infty} je^{-j^2 y/2} &\leq 1 + 3e^{-y/2} + 3 \int_1^{\infty} te^{-t^2 y/2} dt \\
 &= \frac{1}{y} \{3e^{-y/2} + y + 3ye^{-y/2}\} \leq \frac{1}{y} \{3 + 4y\} \leq \frac{3.4}{y}.
 \end{aligned}$$

Thus, for  $n \geq 25$ ,

$$|\tau(n)| x^n \leq x(3.4/y)^8,$$

or

$$\begin{aligned}
 |\tau(n)| &\leq (1 - 1/n)^{-n} \cdot (1 - 1/n)(3.4)^8 \{-\log(1 - 1/n)\}^{-8} \\
 &\leq 2.84(3.4)^8 \{1/n + 1/2n^2 + 1/3n^3 + \dots\}^{-8} \leq 51,000n^8,
 \end{aligned}$$

so we have the result for  $n \geq 25$ . For  $1 \leq n < 25$ , the result follows from the tables of  $\tau(n)$ . We have used the fact that  $(1 - 1/n)^n$  is increasing. We next have

TABLE I.  $\Gamma(s) \cdot \sum \tau(n)n^{-s}$

t	$\sigma = 6.0$		$\sigma = 6.5$	
	Re $\Gamma F$	Im $\Gamma F$	Re $\Gamma F$	Im $\Gamma F$
0.00	95.054741	0.0	241.635216	0.0
0.25	84.897790	42.006402	214.297736	109.802252
0.50	56.861760	74.510262	139.175001	193.401836
0.75	17.608952	90.396027	35.036781	231.495179
1.00	-23.713754	86.667437	- 72.600156	216.603934
1.25	-57.823930	65.064019	- 158.323399	154.660635
1.50	-77.603739	31.426603	- 203.406452	62.899797
1.75	-79.731717	- 5.966438	- 200.276555	- 35.205315
2.00	-65.241565	- 38.591744	- 153.721738	- 116.389995
2.25	-38.925282	- 59.781593	- 78.700101	- 163.541602
2.50	- 7.825106	- 66.178287	4.392353	- 169.519207
2.75	20.707071	- 58.209598	75.468918	- 137.964276
3.00	40.826033	- 39.552406	119.825714	- 81.159823
3.25	49.417013	- 15.834251	131.207861	- 15.838631
3.50	46.472398	6.994358	112.251787	41.665388
3.75	34.630023	24.144535	72.510735	79.501835
4.00	18.111104	32.958140	24.936651	92.601304
4.25	1.427858	33.182355	- 17.985120	82.813237
4.50	- 11.760975	26.588121	- 47.326988	57.184320
4.75	- 19.369256	16.145104	- 59.311148	25.163485
5.00	- 21.063762	5.051909	- 55.227435	- 4.302722
5.25	- 17.961022	- 4.093386	- 40.013489	- 24.958162
5.50	- 12.019054	- 9.790036	- 20.156443	- 34.241392
5.75	- 5.351287	- 11.736869	- 1.642814	- 33.071710
6.00	0.336026	- 10.600628	11.479144	- 24.742209

LEMMA 2. For  $|\sigma - 6| \leq \frac{1}{2}$ ,  $s = \sigma + it$  and  $x \geq 10$ ,

$$|\Gamma(s, x)| \leq 1.87x^{\sigma-1}e^{-x}.$$

*Proof.* From [1, 6.5.32], we have

$$\begin{aligned} |\Gamma(s, x)| &\leq \int_x^\infty e^{-t}t^{\sigma-1} dt \leq \Gamma(\sigma, x) \\ &= x^{\sigma-1}e^{-x} \left[ 1 + \frac{5.5}{x} + \frac{(5.5)(4.5)}{x^2} + \frac{(5.5)(4.5)(3.5)}{x^3} \right. \\ &\quad \left. + \frac{(5.5)(4.5)(3.5)(2.5)}{x^4} + \frac{(5.5)(4.5)(3.5)(2.5)(1.5)}{x^5} + R_5 \right] \end{aligned}$$

where  $|R_5| \leq (5.5)(4.5)(3.5)(2.5)(1.5)(.5)/x^6$ .

t	σ = 6.0		σ = 6.5	
	Re Γ F	Im Γ F	Re Γ F	Im Γ F
6.25	4.060183	- 7.590891	17.659110	- 13.435161
6.50	5.606613	- 4.004048	17.576528	- 2.870248
6.75	5.358784	- 0.868649	13.326154	4.561652
7.00	4.017186	1.23 7244	7.432091	8.055152
7.25	2.315988	2.193480	2.021947	8.125970
7.50	0.817445	2.218105	- 1.641478	6.047054
7.75	- 0.180696	1.692174	- 3.234755	3.237683
8.00	- 0.632869	0.996442	- 3.148548	0.802248
8.25	- 0.665268	0.403751	- 2.121957	- 0.686960
8.50	- 0.473370	0.040814	- 0.892740	- 1.172424
8.75	- 0.235696	- 0.092358	0.032502	- 0.940076
9.00	- 0.064737	- 0.071203	0.454587	- 0.398768
9.25	0.002628	0.009194	0.431195	0.101857
9.50	- 0.013714	0.074908	0.159194	0.368353
9.75	- 0.067839	0.091012	- 0.147665	0.372226
10.00	- 0.115695	0.058884	- 0.342453	0.198228
10.25	- 0.131613	0.001489	- 0.374718	- 0.028725
10.50	- 0.111796	- 0.053749	- 0.274697	- 0.204745
10.75	- 0.068324	- 0.087460	- 0.113729	- 0.278673
11.00	- 0.019278	- 0.093322	0.036463	- 0.253193
11.25	0.019980	- 0.076408	0.130413	- 0.165736
11.50	0.041557	- 0.047840	0.156643	- 0.063219
11.75	0.045459	- 0.019011	0.130285	0.017928
12.00	0.037030	0.002500	0.078600	0.060969

Hence, for  $x \geq 10$ ,

$$|\Gamma(s, x)| \leq x^{\sigma-1} e^{-x} \{1 + .550 + .222 + .071 + .018 + .003 + .002\},$$

and the lemma follows. We next estimate the truncation error for the series (6) and (7).

We have, for  $N \geq 8$ ,

$$\begin{aligned} \left| \sum_{n=N+1}^{\infty} \tau(n)n^{-s} \Gamma(s, n) \right| &\leq \sum_{n=N+1}^{\infty} 51,000n^8 n^{-\sigma} n^{\sigma-1} e^{-n} (1.87) \\ &\leq 9.6 \times 10^4 \sum_{n=N+1}^{\infty} n^7 e^{-n} \\ &\leq 9.6 \times 10^4 \int_N^{\infty} x^7 e^{-x} dx \\ &= 9.6 \times 10^4 e^{-N} \{N^7 + 7N^6 + 7 \cdot 6N^5 + \dots + 7!\} \\ &\leq 3.2 \times 10^5 N^7 e^{-N}, \end{aligned}$$

t	$\sigma = 6.0$		$\sigma = 6.5$	
	Re $\Gamma F$	Im $\Gamma F$	Re $\Gamma F$	Im $\Gamma F$
12.25	0.023387	0.013479	0.027219	0.067722
12.50	0.010481	0.015706	- 0.008103	0.051173
12.75	0.001620	0.012488	- 0.023197	0.026800
13.00	- 0.002558	0.007458	- 0.022402	0.006247
13.25	- 0.003192	0.003163	- 0.013631	- 0.005196
13.50	- 0.002047	0.000682	- 0.004077	- 0.007832
13.75	- 0.000630	- 0.000088	0.002079	- 0.005069
14.00	0.000232	0.000158	0.003914	- 0.000801
14.25	0.000377	0.000668	0.002673	0.002293
14.50	0.000052	0.000961	0.000318	0.003261
14.75	- 0.000384	0.000896	- 0.001586	0.002497
15.00	- 0.000671	0.000572	- 0.002344	0.000984
15.25	- 0.000717	0.000181	- 0.002045	- 0.000375
15.50	- 0.000570	- 0.000123	- 0.001193	- 0.001102
15.75	- 0.000341	- 0.000272	- 0.000318	- 0.001170
16.00	- 0.000131	- 0.000279	0.000254	- 0.000828

so  $N \geq 70$  will give a truncation estimate of  $< 10^{-10}$ . Using  $|\tau(n)n^{-\sigma}| \leq 2$  in this range, we can take  $N \geq 51$ . For (7), we estimate

$$\begin{aligned}
 \left| \sum_{n=N+1}^{\infty} \tau(n)n^{s-12}\Gamma(12-s, 4\pi^2 n) \right| &\leq \sum_{n=N+1}^{\infty} 51,000n^8 n^{\sigma-12}(1.87)(4\pi^2 n)^{11-\sigma} e^{-4\pi^2 n} \\
 &\leq 9.6 \times 10^4 \sum_{n=N+1}^{\infty} n^7 e^{-4\pi^2 n} (4\pi^2)^{11-\sigma} \\
 &\leq 9.6 \times 10^4 (4\pi^2)^{5.5} \int_N^{\infty} x^7 e^{-4\pi^2 x} dx \\
 &= 9.6 \times 10^4 (4\pi^2)^{5.5-8} \int_{4\pi^2 N}^{\infty} u^7 e^{-u} du \\
 &\leq 3.2 \times 10^5 (4\pi^2)^{5.5-8+7} N^7 e^{-4\pi^2 N} \\
 &\leq 5 \times 10^{12} N^7 e^{-4\pi^2 N}
 \end{aligned}$$

which is  $\leq 10^{-10}$  for  $N \geq 4$ . Thus, for this series, we need only a very few terms.

For convenience in the calculation, one calculates the real and imaginary parts of

$$\int_n^{n+1} u^{s-1} e^{-u} du, \quad n = 1, 2, \dots,$$

using Simpson's rule, and sums the results to get  $\Gamma(s, n)$ ,  $n = 1, 2, \dots$ . The number of steps for each integration varies directly as  $t$ , and also the  $\Gamma$  factor shrinks the size of the answer as  $t$  increases, so the method is only suitable for a restricted range of  $t$ .

Table I gives the rounded real and imaginary parts of  $\Gamma(s)F(s)$  for  $\sigma = 6.0, 6.5$

and  $t = 0(.25)16$ . Two zeros  $6 + iz_i$  were found in this range

$$z_1 = 9.22238, \quad z_2 = 13.90755,$$

and the change of argument around the rectangle with vertices  $5.5, 6.5, 6.5 + 16i, 5.5 + 16i$  indicates that these are the only zeros in this region. A rigorous machine proof could be given by taking a mesh sufficiently fine. The third zero appears to be

$$z_3 = 17.443.$$

In principle, the method can be used as high up in the critical strip as one desires, but in practice, the problem is that the calculation of the incomplete  $\Gamma$ -function starts to take too much time. J. B. Rosser [5] studied this problem. Several other possible methods of calculation were studied. What one would like is to take a number of terms of  $\sum \tau(n)n^{-s}$  and find a correction term, or to use some forms of the approximate functional equation (Apostol and Sklar [3]). The present author could not see how to do it.

We close this paper with the following

**THEOREM.** For  $6 < \sigma < 6.5, |t| \geq 4.35$ , if  $F(s) \neq 0$ , then

$$|F(12 - s)| > |F(s)|.$$

*Proof.* We proceed as in Schoenfeld and Dixon [4]. Write, from (2) and (4),

$$F(12 - s) = f(s)F(s),$$

$$f(s) = (2\pi)^{2(6-s)} \Gamma(s)/\Gamma(12 - s).$$

Taking  $s_0 = 6 + it$ , we have  $|f(s_0)| = 1$  for all  $t$ . Setting  $h(s) = \log |g(s)/g(s_0)|$ , it suffices to prove  $h(s) > 0$  for  $6 < \sigma < 6.5$  and  $t \geq 4.35$ . We have

$$h(s) = \log \left| \frac{(2\pi)^{2(6-s)} \Gamma(s)}{(2\pi)^{2it}} \frac{\Gamma(6 + it)}{\Gamma(12 - s)} \frac{\Gamma(6 - it)}{\Gamma(6 - it)} \right|$$

$$= -2(\sigma - 6) \log 2\pi + \log |\Gamma(s)| - \log |\Gamma(12 - \bar{s})|$$

$$= -2(\sigma - 6) \log 2\pi + 2(\sigma - 6)(\partial/\partial\sigma) \log |\Gamma(\sigma + it)|_{\sigma=\sigma_1}$$

where  $5.5 \leq \sigma_1 \leq 6.5$ . Thus

$$\frac{h(s)}{2(\sigma - 6)} = \frac{\partial}{\partial\sigma} \log |\Gamma(\sigma + it)|_{\sigma=\sigma_1} - \log 2\pi,$$

and, as in [4],

$$\frac{\partial}{\partial\sigma} \log |\Gamma(\sigma + it)| \geq \log |s| - \frac{1}{2|s|} - \frac{1}{12|s|^2} - \frac{1}{8} \int_0^\infty \frac{dx}{\{(\sigma + x)^2 + t^2\}^2}$$

$$\geq \log |5.5 + it| - \frac{1}{14} - \frac{1}{12 \cdot 7^2} - \frac{1}{8} \int_0^\infty \frac{dx}{(5.5 + x)^2}$$

$$= \log |5.5 + it| - \frac{1}{12} - \frac{1}{588} - \frac{1}{44}.$$

Thus, the result holds if

$$\log |5.5 + it| > \log 2\pi + .108$$

or if  $\log |5.5 + it| > \log 7$ , or  $t^2 > 18.75$ , or  $t \geq 4.35$ , and the proof is complete. Berndt [6] proved the result for  $|t| \geq 6.8$ ,  $\sigma > 6$ . Empirically, for  $\sigma = 6.5$ , the inequality appears to fail for  $0 \leq t \leq 3$  and appears to hold for  $t \geq 3.25$ .

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