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Saddle Points of the Complementary Error Function

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Abstract. The first one hundred zeros of the derivative of the function $w(z) = e^{-z^2}$ Erfc$(-iz)$ are given, together with an asymptotic formula for estimating the higher zeros.

1. In a previous paper by the present authors [1], the zeros of the function

$$w(z) = e^{-z^2} \operatorname{Erfc}(-iz) \tag{1}$$

were obtained. In this paper, the values of $z = x + iy$ for which

$$dw/dz = 0 \tag{2}$$

are given. These points represent singular points of the family of curves

$$\phi(x, y) \equiv |w| = \text{const} \tag{3}$$

in the x-y plane since at such a point the direction dy/dx of these curves is undefined. As in the case of the zeros of $w(z)$, the saddle points lie in the lower half-plane and are symmetrically located with respect to the y-axis. For convenience, we introduce the function $Y(\rho) = (\sqrt{\pi}/2)w(i\rho)$, which satisfies the differential equation

$$dY/d\rho = 2\rho Y - 1. \tag{4}$$

Thus, at a saddle point, $\rho = \rho_n$,

$$2\rho_n Y(\rho_n) = 1. \tag{5}$$

With the aid of the differential equation (4), we can expand Y in the vicinity of a saddle point as a Taylor series, viz.,

$$Y = +\frac{1}{2\rho_n} + \frac{1}{2\rho_n}(\rho - \rho_n)^2 + \tfrac{1}{3}(\rho - \rho_n)^3 + \cdots . \tag{6}$$

Hence

$$\frac{1}{Y} = 2\rho_n - 2\rho_n(\rho - \rho_n)^2 - \frac{4\rho_n^2}{3}(\rho - \rho_n)^3 + \cdots . \tag{7}$$

Introducing the variable $t = \rho - 1/2Y$, this may be written

$$\begin{aligned} t &= (\rho - \rho_n) + \rho_n(\rho - \rho_n)^2 + \frac{2\rho_n^2}{3}(\rho - \rho_n)^3 + \cdots \\ &= (\rho - \rho_n) + \rho(\rho - \rho_n)^2 - \left[1 - \frac{2\rho^2}{3}\right](\rho - \rho_n)^3 + \cdots . \end{aligned} \tag{8}$$

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Therefore

$$\rho - \rho_n = t - \rho t^2 + [1 - 8\rho^2/3]t^3 + \cdots , \tag{9}$$

or

$$\rho_n = \rho - t + \rho t^2 - [1 - 8\rho^2/3]t^3 + \cdots . \tag{10}$$

Equation (3) may also be expressed in terms of Y as follows:

$$\rho_n = \frac{1}{2Y}\left[1 + t^2 + \frac{4}{3Y}\, t^3 + \cdots\right]. \tag{11}$$

The above series will converge rapidly if ρ is close to a saddle point ρ_n. In the next section, an asymptotic approximation to the saddle points is derived which may be used as a first approximation. By computing the corresponding values of Y and t and substituting these into Eq. (11), an improved approximation to ρ_n is obtained. If necessary,* the process may be repeated using the newly computed value of ρ, and continued until convergence is reached. A sample calculation leading to the first saddle point is given at the end of the next section.

2. Asymptotic Approximation to the Saddle Points. At a saddle point, we have, from Eq. (5), $2\rho Y = 1$ or

$$w = +i/\pi^{1/2}z. \tag{12}$$

The saddle points are assumed to be of the form $z = x - iy$, with $x > 0$, $y > 0$. Setting $w(x + iy) = u + iv$, Eq. (12) is equivalent to

$$2e^{y^2-x^2}e^{2ixy} - u + iv = i/\pi^{1/2}z. \tag{13}$$

Replacing w by the first three terms of the continued fraction gives

$$u - iv = -\frac{i}{\pi^{1/2}}\left[\frac{z^2 - 1}{z(z^2 - 3/2)}\right], \tag{14}$$

and Eq. (13) becomes

$$2e^{y^2-x^2}e^{2ixy} \doteq \frac{-i}{\pi^{1/2}}\left\{\frac{1}{z[2z^2 - 3]}\right\}. \tag{15}$$

Since $\arg(z) = -\pi/4 + \sigma$, it follows that the argument of the right side of (15) is $\pi/4 - \sigma$. Hence,

$$2xy = (2n + \tfrac{1}{4})\pi + \beta, \tag{16}$$

where $0 \leqq \beta \leqq \pi/2$ and since, asymptotically, $x \doteq y$, we take, as the limiting value of x and y,

$$\lambda = ((n + \tfrac{1}{8})\pi)^{1/2} \tag{17}$$

and set**

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* By computing a sufficient number of additional terms in Eq. (11), only one application would be required.

** For the justification of this form, see [1, Eq. (29)].

(18) $$x = \lambda + \alpha + p, \qquad y = \lambda - \alpha + p.$$

From Eq. (16) we have, equating magnitudes,

(19) $$2e^{-4\lambda\alpha - 4\alpha p} \doteq \frac{1}{2\sqrt{\pi}\,(x^2 + y^2)^{3/2}} = \frac{1}{2^{5/2}\sqrt{\pi}\,[\lambda^2 + \alpha^2 + 2\lambda p]^{3/2}} = \frac{1}{2^{5/2}\sqrt{\pi}\,\lambda^3[1 + 2p/\lambda + (\alpha/\lambda)^2]^{3/2}}.$$

Hence

(20) $$2e^{-4\lambda\alpha} = 1/2^{5/2}\pi^{1/2}\,\lambda^3;$$

(21) $$\alpha \doteq \ln(128\pi\lambda^6)/8\lambda.$$

The value of p is determined by equating arguments in Eq. (15). We find, denoting the argument of the right side by ϕ,

(22) $$\tan 2xy \doteq 1 + 4\alpha^2 - 8\lambda p;$$

(23) $$\tan\phi \doteq 1 - 6\alpha/\lambda + 3/2\lambda^2.$$

This gives

(24) $$p \doteq (8(\lambda\alpha)^2 - 12(\lambda\alpha) + 3)/16\lambda^3.$$

Thus, the desired asymptotic approximation to three terms is

(25) $$\begin{Bmatrix} x \\ -y \end{Bmatrix} = \lambda \pm \frac{1}{8\lambda}\ln(128\pi\lambda^6) + \frac{\frac{1}{8}[\ln(128\pi\lambda^6)]^2 - \frac{3}{2}\ln(128\pi\lambda^6) + 3}{16\lambda^3}.$$

The use of the approximation (25) in conjunction with Eq. (11) is illustrated below for the first saddle point. Equation (17) with $n = 1$ gives

(26) $$\lambda = 1.8799712060$$

and this, when substituted into Eq. (25), gives

(27) $$x \doteq 2.5332619139, \qquad y \doteq -1.2321384069.$$

The corresponding value of Y is

(28) $$Y = -.0766358650 + .1594090127i.$$

Thus

(29) $$t = -.0073085147 + .0144867658i.$$

Substituting in Eq. (11) the values of t and y as given by Eqs. (28) and (29), we arrived at the improved values

(30) $$x \doteq 2.5471305433, \qquad y \doteq -1.2251557198,$$

the corresponding values of Y and t being

(31) $$Y = -.07667898752 + .1594172691i$$
$$t = .00000137615 - .00000251508i.$$

Zeros of $w'(z)$

N	X	Y	N	X	Y
1	2.5471280282E+00	-1.2251570959E+00	51	1.2883628281E+01	-1.2464770826E+01
2	3.1619390531E+00	-2.0255961307E+00	52	1.3005504374E+01	-1.2589548699E+01
3	3.6559721638E+00	-2.6288721547E+00	53	1.3126239378E+01	-1.2713113798E+01
4	4.0833844384E+00	-3.1323514518E+00	54	1.3245864838E+01	-1.2835500701E+01
5	4.4663013869E+00	-3.5728492033E+00	55	1.3364410869E+01	-1.2956742377E+01
6	4.8165949556E+00	-3.9690173288E+00	56	1.3481906248E+01	-1.3076870288E+01
7	5.1415732873E+00	-4.3318395797E+00	57	1.3598378492E+01	-1.3195914484E+01
8	5.4461347735E+00	-4.6684220832E+00	58	1.3713853935E+01	-1.3313903689E+01
9	5.7337608049E+00	-4.9836711350E+00	59	1.3828357802E+01	-1.3430865384E+01
10	6.0070346327E+00	-5.2811402113E+00	60	1.3941914268E+01	-1.3546825878E+01
11	6.2679376653E+00	-5.5634993766E+00	61	1.4054546519E+01	-1.3661810379E+01
12	6.5180302553E+00	-5.8328143120E+00	62	1.4166276813E+01	-1.3775843054E+01
13	6.7585678528E+00	-6.0907215155E+00	63	1.4277126525E+01	-1.3888947093E+01
14	6.9905787939E+00	-6.3385432717E+00	64	1.4387116197E+01	-1.4001144760E+01
15	7.2149182067E+00	-6.5773658881E+00	65	1.4496265587E+01	-1.4112457444E+01
16	7.4323064525E+00	-6.8080945823E+00	66	1.4604593703E+01	-1.4222905709E+01
17	7.6433572151E+00	-7.0314930011E+00	67	1.4712118848E+01	-1.4332509334E+01
18	7.8485984629E+00	-7.2482123176E+00	68	1.4818858656E+01	-1.4441287357E+01
19	8.0484883808E+00	-7.4588130776E+00	69	1.4924830120E+01	-1.4549258111E+01
20	8.2434276772E+00	-7.6637818847E+00	70	1.5030049632E+01	-1.4656439265E+01
21	8.4337692264E+00	-7.8635443393E+00	71	1.5134533006E+01	-1.4762847849E+01
22	8.6198257218E+00	-8.0584752099E+00	72	1.5238295510E+01	-1.4868500294E+01
23	8.8018758200E+00	-8.2489065255E+00	73	1.5341351892E+01	-1.4973412456E+01
24	8.9801691250E+00	-8.4351340870E+00	74	1.5443716400E+01	-1.5077599645E+01
25	9.1549302694E+00	-8.6174227572E+00	75	1.5545402812E+01	-1.5181076652E+01
26	9.3263622857E+00	-8.7960108003E+00	76	1.5646424452E+01	-1.5283857772E+01
27	9.4946494133E+00	-8.9711134684E+00	77	1.5746794214E+01	-1.5385956828E+01
28	9.6599594522E+00	-9.1429259916E+00	78	1.5846524580E+01	-1.5487387190E+01
29	9.8224457500E+00	-9.3116260862E+00	79	1.5945627637E+01	-1.5588161800E+01
30	9.9822488903E+00	-9.4773760740E+00	80	1.6044115098E+01	-1.5688293186E+01
31	1.0139498135E+01	-9.6403246831E+00	81	1.6141998312E+01	-1.5787793485E+01
32	1.0294312663E+01	-9.8006085869E+00	82	1.6239288287E+01	-1.5886674456E+01
33	1.0446802641E+01	-9.9583537267E+00	83	1.6335995699E+01	-1.5984947497E+01
34	1.0597070150E+01	-1.0113676453E+01	84	1.6432130905E+01	-1.6082623664E+01
35	1.0745209996E+01	-1.0266684518E+01	85	1.6527703961E+01	-1.6179713679E+01
36	1.0891310412E+01	-1.0417477933E+01	86	1.6622724632E+01	-1.6276227949E+01
37	1.1035453688E+01	-1.0566149730E+01	87	1.6717202401E+01	-1.6372176576E+01
38	1.1177716709E+01	-1.0712786621E+01	88	1.6811146485E+01	-1.6467569372E+01
39	1.1318171446E+01	-1.0857469582E+01	89	1.6904565840E+01	-1.6562415867E+01
40	1.1456885382E+01	-1.1000274370E+01	90	1.6997469177E+01	-1.6656725323E+01
41	1.1593921898E+01	-1.1141271981E+01	91	1.7089864966E+01	-1.6750506743E+01
42	1.1729340610E+01	-1.1280529058E+01	92	1.7181761449E+01	-1.6843758883E+01
43	1.1863197680E+01	-1.1418108258E+01	93	1.7273166646E+01	-1.6936520258E+01
44	1.1995546087E+01	-1.1554068572E+01	94	1.7364088366E+01	-1.7028769153E+01
45	1.2126435874E+01	-1.1688465623E+01	95	1.7454534212E+01	-1.7120523634E+01
46	1.2255914376E+01	-1.1821351925E+01	96	1.7544511589E+01	-1.7211791552E+01
47	1.2384025412E+01	-1.1952777125E+01	97	1.7634027714E+01	-1.7302580552E+01
48	1.2510814479E+01	-1.2082788211E+01	98	1.7723089617E+01	-1.7392898083E+01
49	1.2636318910E+01	-1.2211429714E+01	99	1.7811704155E+01	-1.7482751401E+01
50	1.2760578027E+01	-1.2338743879E+01	100	1.7899878011E+01	-1.7572147579E+01

This leads to the next approximation

(32) $$x \doteq 2.5471280282, \qquad y \doteq -1.2251570959.$$

which is now correct to eleven figures, the error being $O(t^4)$.

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1. HENRY E. FETTIS, JAMES C. CASLIN & KENNETH R. CRAMER, "Complex zeros of the error function and of the complementary error function," *Math. Comp.*, v. 27, 1973, pp. 401–407.