MATHEMATICS OF COMPUTATION, VOLUME 27, NUMBER 122, APRIL, 1973

Bounds on Iterated Coerror Functions and Their Ratios

By D. E. Amos

Abstract. Upper and lower bounds on $y_n = i^n \operatorname{erfc}(x)$ and $r_n = y_n/y_{n-1}$, $n \geqq 1$, $-\infty < x < \infty$, are established in terms of elementary functions. Numerical procedures for refining these bounds are presented so that r_n and y_k, $k = 0, 1, \ldots, n$, can be computed to a specified accuracy. Some relations establishing bounds on r_n' and r_n'' are also derived.

Simple Bounds. Let $y_n(x) = i^n \operatorname{erfc}(x)$, $n = -1, 0, 1, \cdots$. The basic inequality (see (31))

$$P_n(x) = y_{n-2}(x)y_n(x) - y_{n-1}^2(x) < 0, \qquad n \geqq 1, \tag{1}$$

expressing monotone decreasing behavior of $r_n(x) = y_n(x)/y_{n-1}(x)$, $n \geqq 1$, in both n and x,

$$r_n' = \frac{P_n(x)}{y_{n-1}^2(x)} = \frac{r_n - r_{n-1}}{r_{n-1}} < 0, \tag{2}$$

is derived in the Appendix. The utility of this relation follows from the recurrence formulae for the iterated coerror function,

$$\begin{gathered} y_{-1}(x) = i^{-1} \operatorname{erfc}(x) = 2e^{-x^2}/\pi^{1/2}, \qquad y_0(x) = i^0 \operatorname{erfc}(x) = \operatorname{erfc}(x), \\ y_{n-2} = 2ny_n + 2xy_{n-1}, \qquad n = 1, 2, \cdots, \end{gathered} \tag{3}$$

to yield

$$r_n' = 2nr_n^2 + 2xr_n - 1 < 0, \qquad n \geqq 1. \tag{4}$$

This establishes bounds on the ratios $r_n = y_n/y_{n-1}$,

$$0 < r_n < \frac{-x + (x^2 + 2n)^{1/2}}{2n} = B_n(x)$$

since the parabola $v = 2nt^2 + 2xt - 1$ is negative between its roots. The upper bound is of most interest and we write $B_n(x)$ in the form

$$\begin{aligned} B_n(x) &= \frac{-x + (x^2 + 2n)^{1/2}}{2n}, \qquad x < 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad n \geqq 1, \\ &= \frac{1}{x + (x^2 + 2n)^{1/2}}, \qquad x \geqq 0, \end{aligned} \tag{5}$$

Received October 29, 1971, revised May 23, 1972.

AMS (MOS) subject classifications (1970). Primary 65D20, 33A20; Secondary 41-04.

Key words and phrases. Iterated coerror function, error function, coerror function, Mill's ratio, probability integral.

to avoid losses of significance in computation when x is large and positive. Furthermore, if we divide (3) by y_{n-1} and solve for r_{n-1}, we get

$$r_{n-1} = \frac{1}{2x + 2nr_n} > \frac{1}{2x + 2nB_n(x)} = B_n(x), \qquad n \geqq 1. \tag{6}$$

Thus, we have (with a shift of index)

$$C_n(x) < r_n < D_n(x), \tag{7}$$

and recursion on (7) in the form $C_n y_{n-1} < y_n < D_n y_{n-1}$ produces

$$\operatorname{erfc}(x) \prod_{k=1}^{n} C_k(x) < y_n < \operatorname{erfc}(x) \prod_{k=1}^{n} D_k(x) \tag{8}$$

where, in this context,

$$C_k(x) = B_{k+1}(x) \quad \text{and} \quad D_k(x) = B_k(x). \tag{9}$$

Table 1 gives some numerical results for these bounds. Both relative and absolute errors can be assessed. The number of significant digits in y_n is approximately the minimum number in agreement between C_k and D_k as k ranges from 1 to n.

TABLE 1

C_n and D_n of (9) compared with r_n

n\x	-10	-5	0	5	10
1	1.005E+01 1.000E+01 5.050E+00	5.098E+00 5.000E+00 2.596E+00	7.071E-01 5.642E-01 5.000E-01	9.808E-02 9.635E-02 9.629E-02	4.975E-02 4.951E-02 4.951E-02
10	1.048E+00 1.043E+00 9.566E-01	5.854E-01 5.788E-01 5.389E-01	2.236E-01 2.181E-01 2.132E-01	8.541E-02 8.449E-02 8.435E-02	4.772E-02 4.753E-02 4.752E-02
20	5.458E-01 5.438E-01 5.218E-01	3.266E-01 3.240E-01 3.139E-01	1.581E-01 1.562E-01 1.543E-01	7.656E-02 7.598E-02 7.584E-02	4.580E-02 4.564E-02 4.563E-02
50	2.414E-01 2.408E-01 2.374E-01	1.618E-01 1.612E-01 1.595E-01	1.000E-01 9.950E-02 9.901E-02	6.180E-02 6.156E-02 6.146E-02	4.142E-02 4.132E-02 4.130E-02
100	1.366E-01 1.364E-01 1.355E-01	1.000E-01 9.978E-02 9.934E-02	7.071E-02 7.053E-02 7.036E-02	5.000E-02 4.989E-02 4.983E-02	3.660E-02 3.654E-02 3.653E-02
200	8.090E-02 8.082E-02 8.061E-02	6.404E-02 6.396E-02 6.384E-02	5.000E-02 4.994E-02 4.988E-02	3.904E-02 3.899E-02 3.897E-02	3.090E-02 3.087E-02 3.086E-02

For $x \geqq 0$, the upper bound of (7) immediately implies

$$y_n(x) < \frac{y_{n-1}(x)}{x + (x^2 + 2n)^{1/2}} \leqq \frac{y_{n-1}(x)}{(2n)^{1/2}} < y_{n-1}(x), \qquad n \geqq 1, x \geqq 0. \tag{10}$$
$$y_n(x) \leqq \frac{y_{n-1}(x)}{x + (x^2 + 2)^{1/2}},$$

$y_n < y_{n-1}$ can also be established for $n \geqq 0$ by induction on

$$y_n(x) = \int_x^\infty y_{n-1}(t)\, dt, \qquad n \geqq 0, \tag{11}$$

and the upper bound in Mill's ratio [4], [14, p. 343]

$$m(x) = \frac{\pi/2}{(\pi - 1)x + (x^2 + \pi)^{1/2}} \leqq \frac{i^0 \operatorname{erfc}(x)}{i^{-1} \operatorname{erfc}(x)} \tag{12}$$
$$\leqq \frac{\pi/2}{2x + ([(\pi - 2)x]^2 + \pi)^{1/2}} = M(x), \qquad x \geqq 0.$$

(See references for other results on Mill's ratio.) With these inequalities, purely elementary bounds on y_n can be given:

$$\frac{2m(x)e^{-x^2}}{\pi^{1/2}} \leqq y_0 \leqq \frac{2M(x)e^{-x^2}}{\pi^{1/2}}, \qquad x \geqq 0, \tag{13}$$
$$\frac{2m(x)e^{-x^2}}{\pi^{1/2}} \prod_{k=1}^n C_k(x) < y_n < \frac{2M(x)e^{-x^2}}{\pi^{1/2}} \prod_{k=1}^n D_k(x), \qquad x \geqq 0, n \geqq 1.$$

Elementary bounds for $x < 0$ follow from the identity $\operatorname{erfc}(x) = 2 - \operatorname{erfc}(-x)$. An upper bound on $y_n(x)$ for $x \geqq 0$ can be obtained from (11) and (12) by induction

$$y_n(x) \leqq \frac{2}{\pi^{1/2}} [M(x)]^{n+1} e^{-x^2}, \qquad n \geqq 0, x \geqq 0, \tag{14}$$

using $M(t) \leqq M(x)$ for $t \geqq x$. Simple backward recursion in (10), followed by (12), gives complementary forms

$$y_n(x) \leqq \frac{2/\pi^{1/2} M(x) e^{-x^2}}{2^{n/2}(n!)^{1/2}}, \qquad n \geqq 0, x \geqq 0. \tag{15}$$
$$y_n(x) \leqq \frac{2/\pi^{1/2} M(x) e^{-x^2}}{(x + (x^2 + 2)^{1/2})^n},$$

While the bounds in (7) were obtained from $r_n' < 0$ and are best for large positive x, Eq. (36), $r_n' > -1/n$, represents the other extreme, and bounds from this inequality would be best for large negative x. They are (7) with

$$C_n(x) = \frac{-x + (x^2 + 2n - 2)^{1/2}}{2n}, \qquad x < 0, \qquad n \geqq 1. \tag{16}$$
$$= \frac{1 - 1/n}{x + (x^2 + 2n - 2)^{1/2}}, \qquad x \geqq 0,$$
$$D_n(x) = B_n(x).$$

Table 2 shows the results for these bounds.

TABLE 2

C_n and D_n of (16) compared with r_n

n\x	-10	-5	0	5	10
1	1.005E+01	5.098E+00	7.071E-01	9.808E-02	4.975E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	1.000E+01	5.000E+00	0.	0.	0.
10	1.048E+00	5.854E-01	2.236E-01	8.541E-02	4.772E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	1.043E+00	5.779E-01	2.121E-01	7.787E-02	4.314E-02
20	5.458E-01	3.266E-01	1.581E-01	7.656E-02	4.580E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.437E-01	3.234E-01	1.541E-01	7.343E-02	4.368E-02
50	2.414E-01	1.618E-01	1.000E-01	6.180E-02	4.142E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.407E-01	1.609E-01	9.899E-02	6.091E-02	4.071E-02
100	1.366E-01	1.000E-01	7.071E-02	5.000E-02	3.660E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.363E-01	9.967E-02	7.036E-02	4.967E-02	3.631E-02
200	8.090E-02	6.404E-02	5.000E-02	3.904E-02	3.090E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.079E-02	6.392E-02	4.987E-02	3.892E-02	3.079E-02

Bounds on y_n/y_{n-k} can be obtained by bounding each term of

$$y_n/y_{n-k} = r_n r_{n-1} \cdots r_{n-k+1}.$$

Improved Bounds. The simplicity of the previous bounds is appealing; however, they are not very sharp near $x = 0$. The results of this section improve this situation. Relation (35) in the Appendix is

$$r_n' = 2nr_n^2 + 2xr_n - 1 < -r_n^2, \quad x \geqq 0, \qquad n \geqq 1.$$
$$< -r_n^2 \exp\{-x^2\}, \quad x < 0,$$

Following through as before, we get (7) with the bounds

$$C_n(x) = \frac{2n + 2 + \exp\{-x^2\}}{2x\exp\{-x^2\} + (2n+2)/D_{n+1}(x)} > D_{n+1}(x), \qquad x < 0,$$

$$= \frac{2n+3}{(2n+4)x + (2n+2)(x^2+2n+3)^{1/2}}, \qquad x \geqq 0,$$

(17)

$$D_n(x) = \frac{-x + (x^2 + 2n + \exp\{-x^2\})^{1/2}}{2n + \exp\{-x^2\}}, \qquad x < 0,$$

$$= \frac{1}{x + (x^2 + 2n + 1)^{1/2}}, \qquad x \geqq 0,$$

TABLE 3

C_n and D_n of (17) *compared with r_n*

n\x	-10	-5	0	5	10
1	1.005E+01	5.098E+00	5.774E-01	9.717E-02	4.963E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	5.050E+00	2.596E+00	5.590E-01	9.632E-02	4.951E-02
10	1.048E+00	5.854E-01	2.182E-01	8.487E-02	4.762E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	9.566E-01	5.389E-01	2.180E-01	8.443E-02	4.752E-02.
20	5.458E-01	3.266E-01	1.562E-01	7.620E-02	4.572E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.218E-01	3.139E-01	1.561E-01	7.593E-02	4.564E-02
50	2.414E-01	1.618E-01	9.950E-02	6.163E-02	4.136E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.374E-01	1.595E-01	9.950E-02	6.153E-02	4.131E-02
100	1.366E-01	1.000E-01	7.053E-02	4.992E-02	3.656E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.355E-01	9.934E-02	7.053E-02	4.988E-02	3.654E-02
200	8.090E-02	6.404E-02	4.994E-02	3.900E-02	3.088E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.061E-02	6.384E-02	4.994E-02	3.899E-02	3.087E-02

whose derivatives of order 2 or greater are discontinuous at $x = 0$. (The other half of (35) leads to $C_n(x)$ in (9).) Table 3 shows these results for some numerical values. Notice also that for $x < 0$, D_{n+1} is a lower bound on r_n. While these bounds are relatively good at $x = 0$, they can be made sharp by observing that

$$r_n' = 2nr_n^2 + 2xr_n - 1 \gtrless r_n'(0) = 2nr_n^2(0) - 1, \qquad x \gtrless 0 \tag{18}$$

is exact at $x = 0$, where

$$r_n(0) = \frac{\Gamma((n+1)/2)}{2\Gamma(n/2+1)}, \qquad n \geqq 0. \tag{19}$$

The inequalities in (18) follow from (38) which shows that r_n' is monotone increasing in x. The roots give

$$\text{(20a)} \qquad \begin{aligned} D_n(x) &= \frac{-x + (x^2 + 2na_n)^{1/2}}{2n}, \qquad & x < 0, \\ C_n(x) &= \frac{a_n}{x + (x^2 + 2na_n)^{1/2}}, \qquad & x \geqq 0, \end{aligned}$$

where $a_n = 2nr_n^2(0)$ and, with (6),

(20b)
$$C_n(x) = \frac{-x + (x^2 + 2(n+1)a_{n+1})^{1/2}}{2(n+1)a_{n+1}}, \qquad x < 0,$$
$$D_n(x) = \frac{1}{x + (x^2 + 2(n+1)a_{n+1})^{1/2}}, \qquad x \geqq 0.$$

Table 4 shows the partial improvement over Tables 1, 2 and 3. Bounds analogous to those in (10) and (15) can be formed for $x \geqq 0$ by setting $x = 0$ or $n = 1$ in $D_n(x)$ of (20b). The largest lower bound and the smallest upper bound among the formulae listed would be the best over the range of interest in n and x.

Iterative improvements, generating upper and lower bounds at each step, can be made by recurring backward on (6), starting the continued fraction approximants with some C_n and D_n (see also the next section on numerical computations).

The connection between these bounds and Mill's ratio can be established by taking (6) with $n = 1$ and applying the expressions for $C_1(x)$ or $D_1(x)$. (20a) for $x \geqq 0$ gives the upper bound quoted in the NBS handbook [1, p. 298] while (5) for $x \geqq 0$ gives the lower bound in the same reference. (20b) gives Boyd's [4] lower bound (12).

The bounds in (12) are fairly sharp with maximum relative errors of 1.17% and 0.91%, respectively. Best results are obtained with $M(x)$ for $x \leqq 0.85$ and $m(x)$ for $x > 0.85$, with an overall maximum relative error for this combination of about 0.86%.

TABLE 4

C_n and D_n of (20) compared with r_n

n\x	-10	-5	0	5	10
1	1.003E+01	5.063E+00	5.642E-01	9.704E-02	4.961E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	6.416E+00	3.280E+00	5.642E-01	6.287E-02	3.173E-02
10	1.045E+00	5.818E-01	2.181E-01	8.486E-02	4.762E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	9.989E-01	5.605E-01	2.181E-01	8.176E-02	4.550E-02
20	5.448E-01	3.250E-01	1.562E-01	7.619E-02	4.571E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.334E-01	3.200E-01	1.562E-01	7.502E-02	4.476E-02
50	2.411E-01	1.614E-01	9.950E-02	6.163E-02	4.136E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.394E-01	1.606E-01	9.950E-02	6.136E-02	4.107E-02
100	1.365E-01	9.983E-02	7.053E-02	4.992E-02	3.656E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.361E-01	9.967E-02	7.053E-02	4.983E-02	3.646E-02
200	8.085E-02	6.398E-02	4.994E-02	3.900E-02	3.088E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.076E-02	6.394E-02	4.994E-02	3.898E-02	3.085E-02

Numerical Computation of r_n and y_k, $k = 0, 1, \cdots, n$. In [5] and [6], Gautschi shows that forward recursion on (3) is appropriate for $x \leqq 0$ while an iterative backward technique on (6) (which generates continued fraction approximants) is appropriate for $x > 0$ for stability. It would be hard to improve on the simplicity of the forward recursion for $x \leqq 0$ but some improvement is possible for $x > 0$ because the continued fraction approximants are slowly convergent for x close to zero. The results developed above are exploited to get accurate values of r_n so that the ratios

$$r_{k-1} = \frac{.5}{x + kr_k}, \qquad k = n, n-1, \cdots, 1, \quad x > 0, \tag{21}$$

can be computed for the relation

$$y_k = \frac{2e^{-x^2}}{\pi^{1/2}} \prod_{i=0}^{k} r_i, \qquad k = 0, 1, 2, \cdots, n, \quad x > 0. \tag{22}$$

The method which has proved successful in computing r_n is based upon a restatement of (21) with k replaced by $k + 1$,

$$[2(k+1)(r_{k+1}/r_k)r_k + 2x]r_k = 1,$$

in the form

$$r_k = \frac{1}{x + (x^2 + 2(k+1)R_{k+1})^{1/2}}, \qquad R_{k+1} = \frac{r_{k+1}}{r_k}. \tag{23a}$$

Then, with D_k of (20b) as an initial approximation to r_k for $k \geqq n$ (see Table 4), the algorithm becomes

$$r_k^0 = \frac{1}{x + (x^2 + [2(k+1)r_{k+1}(0)]^2)^{1/2}}, \quad k = n, n+1, \cdots, n+\nu, x \geqq 0, \tag{23b}$$

where $r_k(0)$ is defined in (19), and

$$\begin{aligned} R_{k+1}^m &= r_{k+1}^m / r_k^m, & k &= n, n+1, \cdots, n+\nu-m-1, \\ r_k^{m+1} &= \frac{1}{x + (x^2 + 2(k+1)R_{k+1}^m)^{1/2}}, & m &= 0, 1, 2, \cdots, \nu-1. \end{aligned} \tag{23c}$$

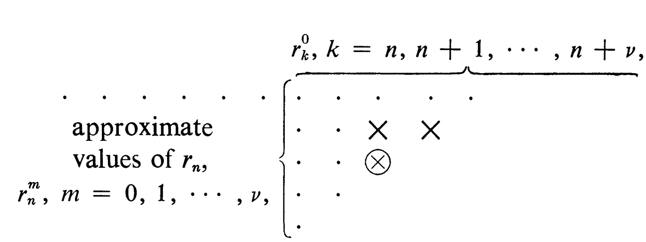

FIGURE 1

The iteration diagram is shown in Fig. 1. r_n^ν is the approximate value of r_n, and only two successive diagonals need be stored.

Although the convergence proof below does not show that r_k^m decreases monotonically to r_k on the columns of Fig. 1, the numerical results were all of this character. If this were true in general, it would follow, on using (21), that

$$c_n^\nu = \frac{1}{2x + 2(n+1)r_{n+1}^{\nu-1}} \leqq r_n \leqq r_n^\nu,$$

and the error criterion

$$|r_n^\nu - r_n|/r_n \leqq |r_n^\nu - c_n^\nu|/c_n^\nu \leqq \epsilon$$

could be used to terminate the process to guarantee a relative error ϵ. Even if the monotonicity is violated, this is a sensible method of termination since $r_{n+1}^{\nu-1}$, and hence c_n^ν, can be expected to be slightly less accurate than r_n^ν. The construction of rigorous bounds using (23a) is shown in the convergence proof of the algorithm.

Some experimentation shows that if $n < 25$ (say) and $x < 2$, n should be increased to 25 before the iteration is started to increase the rate of convergence. Upon completion, the backward recursive step in (21) is applied followed by (22). Notice that for $x = 0$, $r_k^1 = r_k(0)$ and $\nu = 1$. With a relative error $\epsilon = 5 \times 10^{-9}$, extensive evaluation of this procedure showed that $\nu \leqq 5$ (a maximum of 6 applications of (23b) and 15 of (23c)) held for $x \geqq 0$ and $0 \leqq n \leqq 100$. For $n \geqq 50$ the maximum value of ν was 4, but the number of steps in (21) to reduce the index when the starting index is small diminishes this advantage somewhat.

Straight backward recursion with (21) starting with C_{200} or D_{200} of (9) gave only 4 significant digits in erfc(0.1). The corresponding computation with C_{200} or D_{200} of (20a) or (20b), which are accurate at $x = 0$, gave 6 significant figures. This amounts to iteration of these bounds according to (6) or (21). It is common to avoid underflow problems by scaling y_k by e^{x^2} in (22).

The forward recursive loop for $x < 0$ is started with $y_{-1}(x)$ and $y_0(x)$ where $y_0(x) = 2 - \text{erfc}(|x|)$ for $x < 0$. The scheme for $x \geqq 0$ is used to compute erfc($|x|$). If $x < X_0$ ($X_0 = -6$ for a CDC 6600 computer), $y_0(x) = 2$ to the word length of the machine and the erfc ($|x|$) computation can be avoided.

The methods exploited here have concentrated on recursion, primarily for sake of computation. However, the differential inequalities developed in the Appendix can be integrated for other types of apprxoimations.

Convergence of the Algorithm.

THEOREM. *If $x \geqq 0$, the sequence r_k^m generated by* (23c) *converges to r_k as $m \to \infty$ for each $k \geqq n \geqq 0$.*

The proof consists of constructing monotone sequences of upper and lower bounds on r_k^m which converge to r_k. Let $D_k^0 = r_k^0$. Using (20b) and (21), we have

$$C_k^0 = \frac{1}{2x + 2(k+1)D_{k+1}^0} \leqq r_k \leqq r_k^0 = D_k^0. \tag{24}$$

Substitution of these bounds into the expressions r_{k+1}/r_k and r_{k+1}^0/r_k^0 yield

$$\frac{C_{k+1}^0}{D_k^0} \leqq \frac{r_{k+1}}{r_k} \leqq \frac{D_{k+1}^0}{C_k^0} \quad \text{and} \quad \frac{C_{k+1}^0}{D_k^0} \leqq \frac{r_{k+1}^0}{r_k^0} \leqq \frac{D_{k+1}^0}{C_k^0}. \tag{25}$$

Another substitution of the bounds in (25) into the denominators of (23a) and (23c) for $m = 0$ give new bounds D_k^1 and C_k^1,

$$D_k^1 \geqq r_k \geqq C_k^1, \qquad D_k^1 \geqq r_k^1 \geqq C_k^1$$

where

(26)

$$D_k^1 = \frac{1}{x + (x^2 + 2(k+1)C_{k+1}^0/D_k^0)^{1/2}} \quad \text{and} \quad C_k^1 = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}^0/C_k^0)^{1/2}}.$$

Notice that equality holds throughout for $x = 0$. Continuing in this way, we can inductively construct sequences of bounds D_k^m and C_k^m on r_k and r_k^m. However, convergence is obtained by showing monotonicity of each sequence,

$$D_k^{m+1} \leqq D_k^m \quad \text{and} \quad C_k^{m+1} \geqq C_k^m, \qquad m = 0, 1, 2, \cdots .$$

Thus, for $m = 0$, we need to establish $D_k^0 \geqq D_k^1$ and $C_k^1 \geqq C_k^0$ before going on to the induction. $D_k^0 \geqq D_k^1$ follows by showing that

$$g(x) = \frac{C_{k+1}^0}{D_k^0} = \frac{R_{k+3}(0)(x + (x^2 + 2(k+1)R_{k+1}(0))^{1/2})}{\left[2R_{k+3}(0) - \frac{k+2}{k+3}\right]x + \frac{k+2}{k+3}(x^2 + 2(k+3)R_{k+3}(0))^{1/2}}$$

in the denominator of D_k^1 is greater than

(27) $$R_{k+1}(0) = r_{k+1}(0)/r_k(0) = 2(k+1)r_{k+1}^2(0)$$

in the denominator of D_k^0 for $x > 0$. (The last expression in (27) is obtained from (6) with $x = 0$ and C_{k+1}^0 is obtained from (24) by rationalizing the denominator of D_{k+2}^0.) This inequality, $g(x) \geqq R_{k+1}(0)$, can be proved by showing monotone increasing behavior of $g(x)$ together with $g(0) = R_{k+1}(0)$. $g(0) = R_{k+1}(0)$ follows from (25) and (27) and the fact that equality holds for $x = 0$ in (25). The positivity of $g'(x)$,

$$g'(x) = \frac{R_{k+3}(0)}{(x^2 + a)^{1/2}(x^2 + c)^{1/2}} \cdot \left[\frac{(ab - cd)(ab + cd)x^2 + ac(ab^2 - cd^2) + bx(a - c)A(x)}{A(x)(dx + b(x^2 + a)^{1/2})^2}\right],$$

where

$$a = 2(k+3)R_{k+3}(0), \qquad c = 2(k+1)R_{k+1}(0),$$

$$b = (k+2)/(k+3), \qquad d = 2R_{k+3}(0) - b,$$

$$A(x) = ab(x^2 + c)^{1/2} + cd(x^2 + a)^{1/2},$$

will follow if the quantities $a - c$, d, $ab - cd$, $ab^2 - cd^2$ are shown to be positive. A direct application of (33) and (39) for $x = 0$ gives $a > c$ since

$$R_k(x) = 1 + r_k'(x), \qquad r_{k+1}'(x) > r_k'(x) \Rightarrow R_{k+1}(x) > R_k(x).$$

On the other hand, $d > 0$ follows from (33), (35) and (7), with (9), for $x = 0$ since

$$R_{k+3}(0) = 1 + r_{k+3}'(0) \geqq 1 - 2r_{k+3}^2(0) > 1 - \frac{2}{2(k+3)} = \frac{k+2}{k+3} = b$$

implies $d > b > 0$.

In order to deduce the signs of $ab - cd$ and $ab^2 - cd^2$, we first express $R_{k+3}(0)$ in terms of $r_{k+1}(0)$ by means of (27) and (6),

(28) $$R_{k+3}(0) = 2(k+3)r_{k+3}^2(0), \qquad r_{k+3}(0) = \frac{k+2}{k+3}r_{k+1}(0).$$

Then, with (27) and (28),

$$ab - cd = 4br_{k+1}^2(0)[(k+2)^2 + (k+1)^2 - 4(k+1)^2(k+2)r_{k+1}^2(0)],$$

$$ab^2 - cd^2 = 4b^2r_{k+1}^2(0)[(2k+3) - 4(k+1)(k+2)r_{k+1}^2(0)] \cdot [1 + 4(k+1)(k+2)r_{k+1}^2(0)].$$

If we use (19) for $r_{k+1}(0)$ together with $\Gamma(z+1) = z\Gamma(z)$ in the denominator, we get

$$r_{k+1}^2(0) = \frac{1}{(k+1)^2} \frac{\Gamma^2(k/2+1)}{\Gamma^2((k+1)/2)},$$

and the upper bound of [18],

$$n^{1-s} \leqq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leqq (n+s)^{1-s}, \qquad n > 0, 0 \leqq s \leqq 1,$$

with $n = k/2$, $s = 1/2$, suffices to establish the sign of $ab - cd$,

$$ab - cd \geqq 4br_{k+1}^2(0)\left[2k^2 + 6k + 5 - 4(k+2)\left(\frac{k+1}{2}\right)\right] = 4br_{k+1}^2(0) > 0.$$

However, sharper results are needed to show $ab^2 - cd^2 > 0$ for $k \geqq 0$. The results stated in (45) can be applied, for $k \geqq 4$,

$$\frac{\Gamma^2(k/2+1)}{\Gamma^2((k+1)/2)} < \frac{k}{2} + \frac{1}{4} + \frac{1}{16k} - \frac{1}{32k^2} + \frac{48}{5k^3}, \qquad k \geqq 4,$$

and this yields

$$ab^2 - cd^2 > \frac{4b^2r_{k+1}^2(0)}{k+1}\left[\frac{3}{4} - \frac{3}{8k} - \frac{763}{20k^2} - \frac{384}{5k^3}\right][1 + 4(k+1)(k+2)r_{k+1}^2(0)] > 0$$

for $k \geqq 9$. Direct substitution was used to verify $ab^2 - cd^2 > 0$ for $k = 0$ through $k = 9$. Thus, $D_k^0 \geqq D_k^1$ with strict inequality for $x > 0$.

For C_k^1, we take the defining equation (26) and substitute (24) for C_k^0 to get

$$C_k^1 = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}^0(2x + 2(k+1)D_{k+1}^0))^{1/2}} = \frac{1}{2x + 2(k+1)D_{k+1}^0} = C_k^0.$$

To summarize the situation for $m = 0$, we have

(29) $\qquad D_k^0 \geqq D_k^1 \geqq r_k \geqq C_k^1 = C_k^0 \quad \text{and} \quad D_k^0 \geqq D_k^1 \geqq r_k^1 \geqq C_k^1 = C_k^0.$

Now, we repeat the induction steps (24) through (26) for $m = 1$. Thus, (29) applied to r_{k+1}/r_k and r_{k+1}^1/r_k^1 yields

$$\frac{C_{k+1}^1}{D_k^1} \leqq \frac{r_{k+1}}{r_k} \leqq \frac{D_{k+1}^1}{C_k^1} \quad \text{and} \quad \frac{C_{k+1}^1}{D_k^1} \leqq \frac{r_{k+1}^1}{r_k^1} \leqq \frac{D_{k+1}^1}{C_k^1}.$$

These expressions with (23a) and (23c) for $m = 1$ give new bounds D_k^2 and C_k^2,

$$D_k^2 \geqq r_k \geqq C_k^2, \qquad D_k^2 \geqq r_k^2 \geqq C_k^2,$$

where

$$D_k^2 = \frac{1}{x + (x^2 + 2(k+1)C_{k+1}^1/D_k^1)^{1/2}} \quad \text{and} \quad C_k^2 = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}^1/C_k^1)^{1/2}}.$$

Then, by (29),

$$\frac{C^1_{k+1}}{D^1_k} \geqq \frac{C^0_{k+1}}{D^0_k} \Rightarrow D^1_k \geqq D^2_k \quad \text{and} \quad \frac{D^1_{k+1}}{C^1_k} \leqq \frac{D^0_{k+1}}{C^0_k} \Rightarrow C^2_k \geqq C^1_k.$$

Thus, for $m = 1$ we have

$$D^0_k \geqq D^1_k \geqq D^2_k \geqq r_k \geqq C^2_k \geqq C^1_k = C^0_k \quad \text{and} \quad D^0_k \geqq D^1_k \geqq D^2_k \geqq r^2_k \geqq C^2_k \geqq C^1_k = C^0_k$$

with strict inequality in $C^2_k \geqq C^1_k$ for $x > 0$ because $D^0_k > D^1_k$ for $x > 0$.

Continuing in this way, we compute inductively a sequence D^m_k which is bounded and monotone decreasing while C^m_k is bounded and monotone increasing with r^m_k and r_k between these bounds. Each sequence therefore has a limit D_k and C_k such that

$$D_k = \frac{1}{x + (x^2 + 2(k+1)C_{k+1}/D_k)^{1/2}}, \qquad C_k = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}/C_k)^{1/2}}.$$

Solving for each of these radicals and squaring gives

$$(30) \qquad D_k = \frac{1}{2x + 2(k+1)C_{k+1}}, \qquad C_k = \frac{1}{2x + 2(k+1)D_{k+1}}$$

and combining the relations in (30) produces

$$D_k = \cfrac{1}{2x + \cfrac{2(k+1)}{2x + 2(k+2)D_{k+2}}}, \qquad C_k = \cfrac{1}{2x + \cfrac{2(k+1)}{2x + 2(k+2)C_{k+2}}}.$$

Each of these lead to the continued fraction for r_k which can be developed similarly by repeated application of (6). Therefore, $D_k = C_k = r_k$ and r^m_k converges to r_k.

Acknowledgements. The author wishes to acknowledge the assistance of the referee for his part in clarifying the presentation and offering suggestions on several tedious details. The assistance of Mrs. Sharon L. Daniel is also appreciated for the coding of the numerical experiments for a CDC 6600 computer.

Appendix. Some Relations Involving r_n, r'_n and r''_n. We start with the selfadjoint differential equation

$$\frac{d}{dt}(e^{t^2}y'_n) = 2ne^{t^2}y_n, \qquad n \geqq -1,$$

and apply Green's theorem to the relation

$$\int_x^\infty \left[y_{n-1}\frac{d}{dt}(e^{t^2}y'_n) - y_n \frac{d}{dt}(e^{t^2}y'_{n-1}) \right] dt = 2\int_x^\infty e^{t^2} y_n(t)y_{n-1}(t)\, dt$$

to get

$$y_{n-1}y'_n - y_n y'_{n-1} = -2e^{-x^2}\int_x^\infty e^{t^2}y_n(t)y_{n-1}(t)\, dt, \qquad n \geqq 1,$$

or with (11),

$$(31) \qquad y_n(x)y_{n-2}(x) - y^2_{n-1}(x) = -2e^{-x^2}\int_x^\infty e^{t^2}y_n(t)y_{n-1}(t)\, dt < 0, \qquad n \geqq 1.$$

Therefore,

$$(32)\qquad r_n'(x) = \frac{y_n y_{n-2}}{y_{n-1}^2} - 1 = -\frac{2e^{-x^2}}{y_{n-1}^2(x)} \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t)\, dt < 0,$$

$$(33)\qquad r_n'(x) = \frac{r_n}{r_{n-1}} - 1 < 0 \quad \text{and} \quad 0 < \frac{r_n}{r_{n-1}} < 1, \qquad n \geqq 0.$$

The recursion relation (6), reciprocated and differentiated, provides the recursion relation for r_n',

$$(34)\qquad r_{n-1}' = -2r_{n-1}^2(1 + nr_n'),$$

which shows that $r_n' > -2r_n^2$. On the other hand, a direct estimate of (32) using

$$\exp\{t^2 - x^2\} \geqq 1 \quad \text{for} \quad t \geqq x \geqq 0,$$
$$\geqq e^{-x^2} \quad \text{for} \quad t \geqq x \quad \text{and} \quad x < 0,$$

and the differential form of (11) gives

$$(35)\qquad \begin{aligned} -2r_n^2 < r_n' &< -r_n^2, & x &\geqq 0,\\ &< -r_n^2 \exp\{-x^2\}, & x &< 0. \end{aligned}$$

(34) in the form

$$-r_{n-1}'/(2r_{n-1}^2) = 1 + nr_n' > 0$$

also shows that

$$(36)\qquad r_n' > -1/n,$$

which is better for large negative x than (35). The recursion relation for r_n'' follows from (34) by differentiation,

$$(37)\qquad r_{n-1}r_{n-1}'' + 2nr_{n-1}^3 r_n'' = 2(r_{n-1}')^2.$$

The positivity of r_n'' can be obtained from (32) as follows:

$$\begin{aligned} y_{n-1}^3 r_n'' &= y_{n-2}[y_n y_{n-2} - y_{n-1}^2] - y_n[y_{n-1}y_{n-3} - y_{n-2}^2]\\ &= -2y_{n-2}(x)e^{-x^2}\int_x^\infty e^{t^2} y_n(t)y_{n-1}(t)\,dt + 2y_n(x)e^{-x^2}\int_x^\infty e^{t^2} y_{n-1}(t)y_{n-2}(t)\,dt\\ &= 2e^{-x^2}\int_x^\infty e^{t^2} y_{n-1}(t)[y_{n-2}(t)y_n(x) - y_{n-2}(x)y_n(t)]\,dt. \end{aligned}$$

This gives, after factorization,

$$(38)\qquad y_{n-1}^3 r_n'' = 2e^{-x^2}\int_x^\infty e^{t^2} y_{n-1}(t)y_n(t)y_n(x)\left[\frac{1}{r_n(t)r_{n-1}(t)} - \frac{1}{r_n(x)r_{n-1}(x)}\right] dt > 0.$$

This not only implies monotone increasing behavior of r_n' in x, but differentiation of (33) establishes monotone behavior in n as well,

$$r_{n-1}r_n'' + r_{n-1}'r_n' = r_n' - r_{n-1}' > 0$$

or

$$0 > r_n' > r_{n-1}'. \tag{39}$$

Upper bounds on r_n'' can be obtained from (37) using $r_n'' > 0$. $r_n'' > 0$ also establishes

$$r_n'/r_n > r_{n-1}'/r_{n-1} \tag{40}$$

through differentiation of (33),

$$r_n'' = \frac{r_n}{r_{n-1}}\left(\frac{r_n'}{r_n} - \frac{r_{n-1}'}{r_{n-1}}\right)$$

and with (33) again,

$$1/r_{n-1} - 1/r_n > 1/r_{n-2} - 1/r_{n-1}.$$

That is, second differences $\delta^2(1/r_{n-1})$ are negative,

$$\delta^2(1/r_{n-1}) < 0, \qquad n = 2, 3, \cdots . \tag{41}$$

(39) together with (33) also establishes the monotone decreasing behavior of the differences $r_{n-1} - r_n$,

$$0 < r_{n-1} - r_n < (r_{n-2} - r_{n-1})r_{n-1}/r_{n-2} < (r_{n-2} - r_{n-1})$$

and hence

$$\delta^2(r_{n-1}) > 0, \qquad n = 2, 3, \cdots . \tag{42}$$

The expression in (31) is the numerator of

$$\frac{d^2}{dx^2} \ln y_n(x) = \frac{y_n y_{n-2} - y_{n-1}^2}{y_n^2} < 0, \qquad n \geqq 1. \tag{43a}$$

For $n = 0$ this works out to be

$$\frac{d^2}{dx^2} \ln y_0(x) = -\frac{2y_{-1}(x)y_1(x)}{y_0^2(x)} < 0. \tag{43b}$$

Thus we have also established that $w = \ln i^n \operatorname{erfc}(x)$ is concave down for all x and all $n \geqq 0$.

A Sharp Upper Bound on a Gamma Ratio. We start with the asymptotic expansion [1, p. 257]

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi + \frac{1}{12z} - \frac{1}{360z^3} + R$$

for $z > 0$ and estimate R by the next term $|R| \leqq 1/(1260z^5)$. The application of the final results dictated this accuracy. This expression can be applied for $z = x + 1$ and $z = x + \frac{1}{2}$ to yield

$$\begin{aligned} \ln \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} &= \frac{1}{2} \ln x + \left(x + \frac{1}{2}\right) \ln\left(1 + \frac{1}{x}\right) - x \ln\left(1 + \frac{1}{2x}\right) - \frac{1}{2} \\ &+ \frac{1}{12x}\left[\left(1 + \frac{1}{x}\right)^{-1} - \left(1 + \frac{1}{2x}\right)^{-1}\right] \\ &- \frac{1}{360x^3}\left[\left(1 + \frac{1}{x}\right)^{-3} - \left(1 + \frac{1}{2x}\right)^{-3}\right] + R_1 - R_2 \end{aligned} \tag{44}$$

where

$$|R_1| \leqq 1/(1260x^5), \qquad |R_2| \leqq 1/(1260x^5).$$

Now, the following alternating series for $x \geqq 2$ can be used for terms up to and including x^{-4} in (44),

$$\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + R_4\left(\frac{1}{x}\right), \qquad \left|R_4\left(\frac{1}{x}\right)\right| \leqq \frac{1}{4x^4},$$

$$= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + R_5\left(\frac{1}{x}\right), \qquad \left|R_5\left(\frac{1}{x}\right)\right| \leqq \frac{1}{5x^5},$$

$$\left(1+\frac{1}{x}\right)^{-1} = 1 - \frac{1}{x} + \frac{1}{x^2} + \bar{R}_3\left(\frac{1}{x}\right) \qquad \left|\bar{R}_3\left(\frac{1}{x}\right)\right| \leqq \frac{1}{x^3},$$

$$\left(1+\frac{1}{x}\right)^{-3} = 1 + \hat{R}_1\left(\frac{1}{x}\right), \qquad \left|\hat{R}_1\left(\frac{1}{x}\right)\right| \leqq \frac{3}{x}.$$

Then

$$\ln\frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} = \ln x + \frac{1}{4x} - \frac{1}{96x^3} + E$$

where

$$E = R_4\left(\frac{1}{x}\right) + 2xR_5\left(\frac{1}{x}\right) - 2xR_5\left(\frac{1}{2x}\right) + \frac{1}{6x}\left[\bar{R}_3\left(\frac{1}{x}\right) - \bar{R}_3\left(\frac{1}{2x}\right)\right]$$
$$- \frac{1}{180x^3}\left[\hat{R}_1\left(\frac{1}{x}\right) - \hat{R}_1\left(\frac{1}{2x}\right)\right] + 2R_1 - 2R_2$$

and

$$|E| < 1/x^4, \qquad x \geqq 2.$$

For consistency, terms of degree three or less are carried accurately in estimating the exponential

$$\frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} = xe^{\alpha} = x\left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \sum_{i=4}^{\infty}\frac{\alpha^i}{i!}\right)$$
$$< x\left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{1}{4!}\sum_{i=4}^{\infty}\alpha^i\right)$$
$$< x\left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{\alpha^4}{4!\,(1-\alpha)}\right)$$

where

$$\alpha = 1/(4x) - 1/(96x^3) + E, \qquad 0 < \alpha < 3/16, \quad x \geqq 2.$$

Now we expand the powers of α and bound terms of higher order,

$$x^{-j} \leqq x^{-4}/2^{j-4}, \qquad j \geqq 5, x \geqq 2,$$

to obtain

$$\frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} < x\left(1+\frac{1}{4x}+\frac{1}{32x^2}-\frac{1}{128x^3}+\frac{6}{5x^4}\right), \qquad x \geqq 2. \tag{45}$$

This expression is asymptotically correct in all terms except the last.

Applied Mathematics Division
Sandia Laboratories
Albuquerque, New Mexico 87115

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