REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, p. 212.

13 [2.05, 2.30, 2.55, 7].—YUDELL L. LUKE, *On generating Bessel Functions by Use of the Backward Recurrence Formula*, Report ARL 72–0030, Aerospace Research Laboratories, Air Force Systems Command, United Air Force, Wright-Patterson Air Force Base, Ohio, February 1972, iv + 40 pp., 27 cm.

Approximations to the Bessel functions $I_\nu(z)$ and $J_\nu(z)$, which result from J. C. P. Miller's well-known backward recurrence algorithm, are here expressed in terms of hypergeometric functions. It transpires that some of these approximations are identical with certain rational approximations developed elsewhere by the author [1]. The truncation error and the effect of rounding errors are similarly analyzed. Realistic a priori error bounds emerge along with a demonstration that rounding errors in Miller's algorithm are insignificant.

W. G.

1. Y. L. LUKE, *The Special Functions and Their Approximations*, Vols. 1 and 2, Academic Press, New York, 1969.

14 [2.10].—A. H. STROUD, *Approximate Calculation of Multiple Integrals*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1971, xiii + 431 pp., 23 cm. Price $16.50.

The approximate integration of functions of one variable is a subject which today is reasonably well understood, both in its theoretical and practical ramifications, and which is extensively documented in a number of books. The same, unfortunately, cannot be said for integration in higher dimensions. There are several reasons for this. On the theoretical side, one faces the problem of having to cope with an infinite variety of possible regions over which to integrate, in contrast to one dimension, where every connected region is an interval. In addition, there is no theory of orthogonal polynomials in several variables coming to our aid, which would be comparable in simplicity to the well-known one-dimensional theory. On the practical side, one runs up against what R. Bellman refers to as "the curse of dimensionality". The tensor product of a two-point quadrature rule in 100 dimensions calls for $2^{100} \doteq 10^{30}$ function evaluations, a task well beyond the capabilities of even the fastest computers of today. In spite of these formidable difficulties, a good deal of progress has been made, particularly in the last couple of decades. The book under review is the first major attempt of summarizing and codifying current knowledge in the field. The only major omission is S. L. Sobolev's theory of formulas "with a regular boundary layer", which, however, is discussed in a recent survey article by S. Haber [1], and is also expected to be the subject of a forthcoming book by Sobolev.

The book is divided into two parts, entitled "Theory" and "Tables". Part I is concerned with the existence and construction of integration formulas of the form

$$\int_D w(x)f(x)\,dx = \sum_{i=1}^{N} B_i f(v_i) + E[f]$$

and with the estimation of the remainder term $E[f]$. Here, D is a domain in n-dimensional Euclidean space and w a given weight function (often identically equal to 1). Part II contains tables of virtually all known formulas of this type and related computer programs.

After an introductory chapter, the discussion begins in Chapter 2 with the construction of product formulas for special n-dimensional regions. The object, thus, is to approximate the desired integral in the form of a Cartesian product of lower-dimensional integration rules, the component rules often being one-dimensional Gaussian rules with suitable weight functions. Among the regions treated are the n-dimensional cube, the n-simplex, n-dimensional cones, the n-sphere and its surface, and the n-dimensional torus. A more general (original) result shows how a formula for a solid star-like n-dimensional region can be obtained from a formula known for its surface. While the use of product formulas is restricted to rather special (though common) regions, nonproduct formulas must be envisaged if one wants to deal with more general, or even arbitrary, regions in n-space. An account of such formulas is given in Chapter 3, one of the longest in the book, and also the most heterogeneous one, methodologically. There are a number of objectives one can pursue and, correspondingly, a number of more or less ad hoc approaches to achieve them. The first (and perhaps easiest) objective is to construct integration formulas, for arbitrary regions and weight functions, having algebraic degree d and requiring not more than $N = (n + d)!/(n!d!)$ function evaluations. It is shown that this can be done, essentially by solving a system of linear algebraic equations. The results are analogues of Newton-Cotes formulas in one dimension. The author then goes on to describe P. J. Davis' procedure for constructing a special class of such formulas, distinguished by having all coefficients B_i positive and all points v_i contained in the region of interest. The existence of such formulas was proved earlier by Tchakaloff using nonconstructive arguments. The emphasis then shifts to formulas having relatively low degrees and requiring as few points as possible. A great number of these are presented, both for arbitrary and special regions. Their construction involves the solution of certain systems of nonlinear equations by matrix methods, or else is based on the relationship which exists between multivariate orthogonal polynomials and integration formulas. A rudimentary theory (largely due to the author) which explores this relationship is developed in detail. Formulas which have been constructed specifically for two-dimensional regions are also included, notably Radon's fifth-degree seven-point formula. It is proved, quite generally, that a formula of degree $d = 2k$ in n dimensions can have no fewer than $N = (n + k)!/(n!k!)$ points. Similar, but more complicated, results hold for odd degrees. The chapter concludes with a brief discussion of Romberg-type methods for integration over the n-cube. Chapter 4 deals with the extension of formulas to higher dimensions, in particular, with devices, other than product methods, for extending a formula of degree d for the m-cube to a formula of the same degree for the n-cube, where $n > m$. Chapter 5 presents an extensive survey of error estimates. Two kinds of estimates are considered in detail. Both are based on viewing the error $E[f]$ as a linear functional in one function space or another. In the first type of estimate, the functions f are assumed to possess partial derivatives up to a finite order and the spaces considered are typified by the validity of an appropriate Taylor's formula. For such spaces Sard's theory of representing linear functionals can be applied, leading immediately to the desired estimates. The novelty here is that many of the error constants required in these estimates are carefully tabulated and that a number of three-dimensional plots are reproduced which vividly illustrate the behavior of the kernel functions involved. The second type of estimates applies to analytic functions and uses Hilbert space techniques to bound the norm of the error functional. Chapter 6, finally, reviews Monte Carlo and "number-theoretic" methods for integration over the n-cube. These are basically integration rules with all coefficients equal, and points chosen either at random (with uniform distribution over the cube), or in some other fashion designed to reduce the error. This usually involves equidistribution considerations, and it is here where number theory comes in.

Part II begins with a short chapter defining the eighteen regions in n-space for which integration formulas are to be catalogued. The formulas are subsequently listed in Chapter 8, which clearly is the core of the whole work. For each of the eighteen regions there are tabulated formulas of increasing degrees, beginning with degree $d = 1$ and going as high as degree $d = 11$ (and sometimes higher for special two- and three-dimensional regions). Often several formulas are given for the same degree and those, particularly useful in the author's judgment, are identified. With many of the formulas, error constants are tabulated which are useful not only for error estimation, but also for comparison of formulas. References to the original sources are provided with most formulas. Since the points in many of the integration formulas listed are the vertices of certain convex regular n-dimensional polytopes, Chapter 9 provides the coordinates for the respective vertices. Chapter 10, finally, presents a number of FORTRAN programs for selected integration formulas and programs for the evaluation of error constants and Sard's kernel functions. The book concludes with an extensive bibliography containing well over 300 items, two-thirds of them dated 1960 or later. A notable omission is the book by Sobol′ [2], which may have appeared too late for inclusion.

It is an indication of the rapid development of the field that an open question mentioned on p. 100 has since been settled. I. P. Mysovskik͡h and V. I͡a. C͡herniṫsina [3] showed that regions exist for which the three integrals in (3.12-2) vanish simultaneously. The author's remark (p. 100) "By Theorem 3.15-3 it follows that a fifth-degree formula for R_2 cannot be found with six points" is thereby proved invalid.

The book is written clearly, concisely, and to the point. Typographical errors appear to be very few, and only minor inaccuracies have been noted by the reviewer. On p. 58, e.g., it is misleading to define a hyperplane $\mathcal{H}_{N-1}$ of E_N as "an $(N - 1)$-dimensional subspace of E_N". On page 185 the assumption in the hypercircle inequality (Theorem 5.13-1) is misquoted inasmuch as w should be assumed an element of the hyperplane (5.13-1), not of the hypercircle $\Omega_{r,N}$. To state a conjecture in the form of a theorem (Theorem 4.3-1) is a questionable practice, in the reviewer's opinion. Minor blemishes, such as these, however, do not detract from the enormous value of this monograph as a reference work and systematic exposition of present knowledge on the subject. It will prove an invaluable source of information and a dependable guide to all those who are faced with having to come up with numerical answers to multiple-integration problems.

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1. S. Haber, "Numerical evaluation of multiple integrals," *SIAM Rev.*, v. 12, 1970, pp. 481–526.
2. I. M. Sobol', *Multidimensional Quadrature Formulas and Haar Functions*, Izdat. "Nauka", Moscow, 1969. (Russian).
3. I. P. Mysovskikh & V. IA. Chernitsina, "Answer to a question of Radon," *Dokl. Akad. Nauk SSSR*, v. 198, 1971, pp. 537–539. (Russian)

15 [5, 13.05, 13.15].—G. Duvaut & J. L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972, xx + 387 pp., 25 cm. Price 118 Fr.

Important advances in "classical" mathematical physics have been made in the last two decades, due to the consistent application of new techniques in studying partial differential equations. The book under review is a contribution in this direction. For the most part, the text is concerned with providing rigorous proofs of existence and uniqueness theorems for certain classes of partial differential equations of continuum mechanics that have inequalities as boundary conditions. The authors have made some effort to explain the physical meaning of these problems as well as to provide some context for the methods of functional analysis and Sobolev spaces used to solve them. The diverse areas discussed include the equations of plasticity and (linear) elasticity, non-Newtonian (Bingham) fluids, and boundary value problems for Maxwell's equations, among others.

The book consists of seven chapters that can be read independently. In each chapter, various physical problems are formulated in terms of partial differential equations and boundary conditions and then shown to possess "generalized solutions." A reader cannot help but admire the virtuosity of the authors, yet he is left in doubt concerning the deeper aspects and implications of the subject.

A sequel on numerical methods for the problems considered is promised in the near future.

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16 [7].—Ludo K. Frevel, *Evaluation of the Generalized Error Function*, Department of Chemistry, The Johns Hopkins University, Baltimore, Maryland. Ms. of 8 typewritten pp. deposited in the UMT file.

The author tabulates to 5S (unrounded) the "natural" error function

$$϶(x) = \frac{1}{\Gamma(1 + 1/\nu)} \int_0^x e^{-t^\nu} \, dt$$