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Integer Vectors with Interprimed Components*

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Abstract. Vectors are considered whose components are positive integers. Such a vector is called interprimed if the components all contain exactly the same distinct prime factors. A method is provided for estimating the number of such vectors, all of whose components are less than a given bound. These estimates resolve a conjecture of Erdös and Motzkin.

1. Introduction. In [1] Erdös and Motzkin raised the question of counting the number of pairs of integers (a, b), $1 \leqq a \leqq b \leqq x$, such that a and b have the same set of distinct prime factors. It is proposed that one show that this number is asymptotic to cx, for some constant c. A solution was proposed [2] which contains an error. Applying different methods, we will provide a solution to Erdös' problem as well as more general ones.

More precisely, a vector $(a_1, \cdots, a_m)$ with positive integral components will be called *interprimed* iff the a_i all have the same set of distinct prime factors. Letting $F_m(x)$ equal the number of such vectors with $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_m \leqq x$, it can be shown by elementary methods that

$$F_m(x) = c_m x + O(x^{m/(m+1)+\epsilon}), \tag{1.1}$$

for any $\epsilon > 0$, c_m a constant depending on m. The case $m = 2$ is that of Erdös' problem. The details of the proof will be given for the case $m = 2$. Apart from a certain amount of notational complexity the method carries over to the general case.

Though the method is not pursued here, it should be noted that these problems can also be treated by analytic methods. For example, for the case $m = 2$ one considers the function $\Phi(z, w)$ of the two complex variables z, w defined by

$$\Phi(z, w) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs}/r^z s^w \tag{1.2}$$

where $a_{rs} = 1$ if r and s have the same distinct prime factors, and $a_{rs} = 0$ otherwise. The series in (1.2) defines $\Phi(z, w)$ in the region $\operatorname{Re} z > 1$, $\operatorname{Re} w > 1$. It is continuable analytically into the domain $\operatorname{Re}(z + 2w) > 1$, $\operatorname{Re}(2z + w) > 1$, by the identity $\Phi(z, w) = G(z, w)\zeta(z + w)$, where $\zeta(s)$ is the Reimann zeta function and $G(z, w)$ is given by

$$G(z, w) = \prod_p \left(1 + \frac{1}{(p^z - 1)(p^w - 1)}\right)\left(1 - \frac{1}{p^{z+w}}\right).$$

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Then $\sum_{r\leqq x}\sum_{s\leqq x} a_{rs}$ may be approximated by

$$\frac{-1}{4\pi^2}\int_{c-iT_1}^{c+iT_1}\int_{c-iT_2}^{c+iT_2}\Phi(z, w)\frac{x^{z+w}}{zw}\,dz\,dw \tag{1.3}$$

where $c = 1 + (1/\log x)$, and T_1 and T_2 are appropriate functions of x. The desired estimation of $\sum_{r\leqq x}\sum_{s\leqq x} a_{rs}$ results by deforming the contour of integration in (1.3) to where $c = \frac{1}{3} + \epsilon$.

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2. Notations. The notations used throughout this note are listed below:

(1) $\mu(d)$ = the Möbius function,

(2) (u, v) = the greatest common divisor of u and v,

(3) $\{u, v\}$ = the least common multiple of u and v,

(4) $\mathrm{sq}(m)$ = the squarefree part of m, $\mathrm{sq}(1) = 1$ and for $m > 1$,

$$\mathrm{sq}(m) = \prod_{p/m} p$$

is the product of the distinct primes dividing m. More generally, for any integer $\tau \geqq 1$, define

$$\mathrm{sq}(m, \tau) = \prod_{p/m;\, p\nmid\tau} p,$$

that is, the product of the distinct primes which divide m but not τ. Clearly, $\mathrm{sq}(m, 1) = \mathrm{sq}(m)$.

(5) p will always be used as a generic symbol for a prime, and q to denote a squarefree integer.

(6) $Q(z, A)$ = the number of squarefree integers less than or equal z which are divisible by A. Clearly, if A is not squarefree, $Q(z, A) = 0$.

(7) $\nu(A)$ = the number of distinct prime factors of A.

3. Preliminary Estimates. Various estimates which are needed are cumulated in the following sequence of lemmas.

LEMMA 3.1. *For any fixed integer $A \geqq 1$,*

$$\sum_{t\leqq z;\,(t,A)=1} 1 = \prod_{p/A}(1 - 1/p)z + O(2^{\nu(A)})$$

where the $O(2^{\nu(A)})$ is uniform in A.

Proof. We have

$$\begin{aligned}\sum_{t\leqq z;\,(t,A)=1} 1 &= \sum_{t\leqq z}\sum_{d/(t,A)}\mu(d)\\ &= \sum_{d/A}\mu(d)\sum_{t\leqq z;\, t\equiv 0 (\mathrm{mod}\, d)} 1\\ &= \sum_{d/A}\mu(d)z/d + O(1)\\ &= \prod_{p/A}(1 - 1/p)z + O(2^{\nu(A)}).\end{aligned}$$

LEMMA 3.2. *For a fixed squarefree integer* $A \geqq 1$,

$$Q(z, A) = \frac{z}{A} \sum_d \frac{\mu(d)}{d^2} (d, A) + O(2^{\nu(A)}(z/A)^{1/2}) \tag{3.1}$$

where the $O(2^{\nu(A)}(z/A)^{1/2})$ *is uniform in* A.

Proof.

$$\begin{aligned} Q(z, A) &= \sum_{t \leqq z/A;\,(t,A)=1} \sum_{d^2/t} \mu(d) \\ &= \sum_{d \leqq (z/A)^{1/2};\,(d,A)=1} \mu(d) \sum_{t \leqq z/Ad^2;\,(t,A)=1} 1. \end{aligned}$$

It follows from Lemma 3.1 that the inner sum equals

$$\prod_{p/A} (1 - 1/p) \frac{z}{Ad^2} + O(z^{\nu(A)}),$$

where the error estimate is uniform in A. Using this we get

$$\begin{aligned} Q(z, A) &= \sum_{d \leqq (z/A)^{1/2};\,(d,A)=1} \mu(d)\left(\prod_{p/A}\left(1 - \frac{1}{p}\right)\frac{z}{Ad^2} + O(2^{\nu(A)})\right) \\ &= \prod_{p/A}\left(1 - \frac{1}{p}\right)\frac{z}{A} \sum_{d \leqq (z/A)^{1/2};\,(d,A)=1} \frac{\mu(d)}{d^2} + O(2^{\nu(A)}(z/A)^{1/2}). \end{aligned}$$

Since

$$\sum_{d > (z/A)^{1/2};\,(d,A)=1} \mu(d)/d^2 = O((A/z)^{1/2}),$$

this in turn yields

$$Q(z, A) = \prod_{p/A}\left(1 - \frac{1}{p}\right)\frac{z}{A} \sum_{(d,A)=1} \frac{\mu(d)}{d^2} + O(2^{\nu(A)}(z/A)^{1/2}). \tag{3.2}$$

Finally, noting that

$$\begin{aligned} \prod_{p/A}\left(1 - \frac{1}{p}\right) \sum_{(d,A)=1} \frac{\mu(d)}{d^2} &= \prod_{p/A}\left(1 - \frac{1}{p}\right) \prod_{p \nmid A}\left(1 - \frac{1}{p^2}\right) \\ &= \prod_p \left(1 - \frac{(p, A)}{p^2}\right) \\ &= \sum_d \frac{\mu(d)}{d^2} (d, A), \end{aligned}$$

(3.1) is an immediate consequence of (3.2).

LEMMA 3.3. *Let* γ *and* τ *be given positive squarefree integers such that* $(\gamma, \tau) = 1$. *Then, for any* $\epsilon_1 > 0$,

$$\sum_{1 \leqq m \leqq z;\,\mathrm{sq}(m,\tau) \equiv 0 \pmod{\gamma}} (\mathrm{sq}(m, \tau))^{-1} = O\left(\frac{z^{\epsilon_1}}{\gamma^{1+\epsilon_1}} \prod_{p/\gamma\tau} \frac{1}{1 - 1/p^{\epsilon_1}}\right) \tag{3.3}$$

uniformly in γ *and* τ. (*Note that the* O *does depend on* ϵ_1.)

Proof. We have

$$\sum_{1\le m\le z;\, \mathrm{sq}(m,\tau)\equiv 0 (\mathrm{mod}\ \gamma)} (\mathrm{sq}(m,\tau))^{-1} = \sum_{q\le z;\, q\equiv 0(\mathrm{mod}\ \gamma);\,(q,\tau)=1} \frac{1}{q} \sum_{m\le z;\,\mathrm{sq}(m,\tau)=q} 1$$

$$= \sum_{q\le z;\, q\equiv 0(\mathrm{mod}\ \gamma);\,(q,\tau)=1} \frac{1}{q} \sum_{\hat{m}\le z/q;\,\mathrm{sq}(\hat{m},\tau)/q} 1$$

$$\le \sum_{q\le z;\, q\equiv 0(\mathrm{mod}\ \gamma);\,(q,\tau)=1} \frac{1}{q} \sum_{\hat{m}\le z/q;\,\mathrm{sq}(\hat{m},\tau)/q} \left(\frac{z/q}{\hat{m}}\right)^{\epsilon_1}$$

$$\le z^{\epsilon_1} \sum_{q\le z;\, q\equiv 0(\mathrm{mod}\ \gamma);\,(q,\tau)=1} \frac{1}{q^{1+\epsilon_1}} \sum_{\mathrm{sq}(\hat{m},\tau)/q} \frac{1}{\hat{m}^{\epsilon_1}}$$

$$= z^{\epsilon_1} \sum_{q\le z;\, q\equiv 0(\mathrm{mod}\ \gamma);\,(q,\tau)=1} \frac{1}{q^{1+\epsilon_1}} \prod_{p/q} \frac{1}{1-1/p^{\epsilon_1}} \prod_{p/\tau} \frac{1}{1-1/p^{\epsilon_1}}$$

$$\le \frac{z^{\epsilon_1}}{\gamma^{1+\epsilon_1}} \sum_{q;\,(q,\tau)=1} \frac{1}{q^{1+\epsilon_1}} \prod_{p/q} \frac{1}{1-1/p^{\epsilon_1}} \prod_{p/\gamma\tau} \frac{1}{1-1/p^{\epsilon_1}}$$

$$= O\left(\frac{z^{\epsilon_1}}{\gamma^{1+\epsilon_1}} \prod_{p/\gamma\tau} \frac{1}{1-1/p^{\epsilon_1}}\right)$$

as asserted in the lemma.

The above lemma enables us to obtain

LEMMA 3.4. *Let B and τ be squarefree integers such that $(B, \tau) = 1$. For any given $\epsilon_2 > 0$, there exists a $c = c(\epsilon_2)$, $c > 1$, such that*

$$\sum_{1\le m\le z} \{B, \mathrm{sq}(m,\tau)\}^{-1} = O\left(\frac{c^{\nu(B)}}{B} \prod_{p/\tau} \frac{1}{1-1/p^{\epsilon_2}} z^{\epsilon_2}\right) \tag{3.4}$$

uniformly in B and τ.

Proof. We have

$$\sum_{1\le m\le z} \{B, \mathrm{sq}(m,\tau)\}^{-1} = \frac{1}{B} \sum_{1\le m\le z} \frac{(B, \mathrm{sq}(m,\tau))}{\mathrm{sq}(m,\tau)}$$

$$\le \frac{1}{B} \sum_{\gamma/B} \gamma \sum_{1\le m\le z;\,\mathrm{sq}(m,\tau)\equiv 0(\mathrm{mod}\ \gamma)} \frac{1}{\mathrm{sq}(m,\tau)}.$$

Noting that γ will be squarefree and $(\gamma, \tau) = 1$, we apply Lemma 3.3 to the inner sum, so that the above is O of

$$\frac{z^{\epsilon_2}}{B} \sum_{\gamma/B} \frac{1}{\gamma^{\epsilon_2}} \prod_{p/\gamma\tau} \frac{1}{1-1/p^{\epsilon_2}} \le \frac{z^{\epsilon_2}}{B} \sum_{\gamma/B} \prod_{p/\gamma} \frac{1}{p^{\epsilon_2}-1} \prod_{p/\tau} \frac{1}{1-1/p^{\epsilon_2}}$$

$$\le \frac{z^{\epsilon_2}}{B} \sum_{\gamma/B} c_1^{\nu(\gamma)} \prod_{p/\tau} \frac{1}{1-1/p^{\epsilon_2}}$$

$$\le \frac{z^{\epsilon_2}}{B} (c_1+1)^{\nu(B)} \prod_{p/\tau} \frac{1}{1-1/p^{\epsilon_2}}$$

(where $c_1 = (2^{\epsilon_2} - 1)^{-1}$), and taking $c = c_1 + 1$ yields (3.4).

LEMMA 3.5. *Let B and τ be squarefree integers. For any given $\epsilon_3 > 0$ there exists a constant $c = c(\epsilon_3)$ such that*

$$(3.5)\qquad \sum_{1\leqq m\leqq z;\,\{B,\,\mathrm{sq}(m)\}\equiv 0 \pmod{\tau}} \{B, \mathrm{sq}(m)\}^{-1} = O\left(\frac{z^{\epsilon_3}}{\hat{B}\tau} c^{\nu(\hat{B})} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}}\right)$$

uniformly in B and τ, where $\hat{B} = B(B, \tau)^{-1}$.

Proof. We have

$$\sum_{1\leqq m\leqq z;\,\{B,\,\mathrm{sq}(m)\}\equiv 0 \pmod{\tau}} \{B, \mathrm{sq}(m)\}^{-1} \leqq \frac{1}{\tau} \sum_{1\leqq m\leqq z} \{\hat{B}, \mathrm{sq}(m, \tau)\}^{-1}.$$

Since B is squarefree, $(\hat{B}, \tau) = 1$, and applying Lemma 3.4 to the above yields (3.5).

Note that if $B = \mathrm{sq}(l)$ for some integer l, then

$$\hat{B} = B(B, \tau)^{-1} = \mathrm{sq}(l, \tau)$$

and Lemma 3.5 yields

$$(3.6)\qquad \sum_{1\leqq m\leqq z;\,\{\mathrm{sq}(l),\,\mathrm{sq}(m)\}\equiv 0 \pmod{\tau}} \{\mathrm{sq}(l), \mathrm{sq}(m)\}^{-1} = O\left(\frac{z^{\epsilon_3}}{\tau\, \mathrm{sq}(l, \tau)} c^{\nu(\mathrm{sq}(l,\tau))} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}}\right).$$

Setting $z = l = m_2$, $m = m_1$, $\tau = 1$ in (3.6) and summing over all $m_2 \leqq z$ we obtain

$$\sum_{1\leqq m_1\leqq m_2\leqq z} \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}^{-1} = O\left(z^{\epsilon_3} \sum_{m_2\leqq z} \frac{c^{\nu(\mathrm{sq}(m_2))}}{\mathrm{sq}(m_2)}\right).$$

But it is well known [3] that, for every $\epsilon > 0$, $c^{\nu(t)} = O(t^{\epsilon})$ so that the above is

$$O\left(z^{\epsilon_3+\epsilon} \sum_{m_2\leqq z} \frac{1}{\mathrm{sq}(m_2)}\right).$$

Finally, from (3.3) with $\tau = \gamma = 1$, $\sum_{m_2\leqq z} 1/\mathrm{sq}(m_2) = O(z^{\epsilon_1})$ so that

$$(3.7)\qquad \sum_{1\leqq m_1\leqq m_2\leqq z} \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}^{-1} = O(z^{\epsilon_5}),$$

where $\epsilon_5 = \epsilon_1 + \epsilon_3 + \epsilon$.

4. The Main Lemma. The results of the previous section are next applied to prove

LEMMA 4.1. *Let $V \geqq U$ be given positive numbers, and let $S(U, V, d)$ denote the sum*

$$(4.1)\qquad \sum_{1\leqq m_1\leqq m_2\leqq V;\, m_2>U} \frac{(d, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}}.$$

Then, for any given $\epsilon_4 > 0$,

$$(4.2)\qquad \sum_{d} \frac{\mu(d)}{d^2} S(U, V, d) = O(U^{-1+\epsilon_4})$$

uniformly in V.

Proof. We have

$$S(U, V, d) \leqq \sum_{U<m_2\leqq V} \frac{1}{m_2} \sum_{\tau/d} \tau \sum_{1\leqq m_1\leqq m_2;\,\{\mathrm{sq}(m_1),\mathrm{sq}(m_2)\}\equiv 0 (\mathrm{mod}\ \tau)} \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}^{-1},$$

and we apply (3.6) to the inner sum on the right, with $z = l = m_2$. This yields that $S(U, V, d)$ is O of

$$\sum_{U<m_2\leqq V} \frac{1}{m_2^{1-\epsilon_3}} \sum_{\tau/d} \frac{c^{\nu(\mathrm{sq}(m_2,\tau))}}{\mathrm{sq}(m_2, \tau)} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}}$$

$$\leqq \sum_{\tau/d} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}} \sum_{m_2>U} \frac{c^{\nu(\mathrm{sq}(m_2,\tau))}}{m_2^{1-\epsilon_3}\,\mathrm{sq}(m_2, \tau)} \tag{4.3}$$

$$= \sum_{\tau/d} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}} \sum_{(q,\tau)=1} \frac{c^{\nu(q)}}{q} \sum_{m_2>U;\,\mathrm{sq}(m_2,\tau)=q} \frac{1}{m_2^{1-\epsilon_3}}.$$

But the inner sum

$$\sum_{m_2>U;\,\mathrm{sq}(m_2,\tau)=q} \frac{1}{m_2^{1-\epsilon_3}} = \frac{1}{q^{1-\epsilon_3}} \sum_{\hat{m}_2>U/q;\,\mathrm{sq}(\hat{m}_2,\tau)/q} \frac{1}{\hat{m}_2^{1-\epsilon_3}}$$

$$\leqq \frac{1}{q^{1-\epsilon_3}} \sum_{\hat{m}_2>U/q;\,\mathrm{sq}(\hat{m}_2,\tau)/q} \frac{1}{\hat{m}_2^{1-\epsilon_3}} \left(\frac{\hat{m}_2 q}{U}\right)^{1-2\epsilon_3}$$

(we assume $\epsilon_3 < \frac{1}{2}$), and this in turn is

$$\leqq \frac{1}{q^{\epsilon_3}} \cdot \frac{1}{U^{1-2\epsilon_3}} \prod_{p/q;\,p\nmid\tau} \frac{1}{1 - 1/p^{\epsilon_3}} \prod_{p/\tau} \frac{1}{1 - 1/p^{\epsilon_3}}.$$

Inserting this in (4.3), we have that $S(U, V, d)$ is O of

$$\sum_{\tau/d} \prod_{p/\tau} \left(\frac{1}{1 - 1/p^{\epsilon_3}}\right)^2 \sum_{(q,\tau)=1} \frac{c_2^{\nu(q)}}{q^{1+\epsilon_3}}\, U^{-1+2\epsilon_3}. \tag{4.4}$$

From this it follows that

$$\sum_d \frac{\mu(d)}{d^2} S(U, V, d) = O\left(U^{-1+2\epsilon_3} \sum_d \frac{\mu^2(d)}{d^2} \sum_{\tau/d} \prod_{p/\tau} \left(1 - \frac{1}{p^{\epsilon_3}}\right)^{-2} \sum_{(q,\tau)=1} \frac{c_2^{\nu(q)}}{q^{1+\epsilon_3}}\right).$$

Noting that

$$\sum_{(q,\tau)=1} \frac{c_2^{\nu(q)}}{q^{1+\epsilon_3}} = O\left(\prod_p \left(1 + \frac{c_2}{p^{1+\epsilon_3}}\right)^{-1}\right),$$

this in turn is

$$O\left(U^{-1+2\epsilon_3} \prod_p \left(1 + \frac{1}{p^2}\left(1 + \left(1 - \frac{1}{p^{\epsilon_3}}\right)^{-2}\right)\left(1 + \frac{c_2}{p^{1+\epsilon_3}}\right)^{-1}\right)\right) = O(U^{-1+2\epsilon_3}).$$

5. Elementary Proof of (3.1). We will give the details of the proof for the case $m = 2$. The general case is completely analogous.

If we consider the integers $\leqq x$ such that the distinct primes which divide them are precisely those which divide the squarefree integer q, these are precisely the integers of the form $qm \leqq x$ such that $\mathrm{sq}(m)$ divides q. Thus we have

$$F_2(x) = \sum_{q \leq x} \sum_{1 \leq m_1 q \leq m_2 q \leq x;\, \mathrm{sq}(m_1)/q;\, \mathrm{sq}(m_2)/q} 1$$
$$= \sum_{1 \leq m_1 \leq m_2 \leq x} \sum_{q \leq x/m_2;\, q \equiv 0 (\mathrm{mod}\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})} 1$$

or

$$F_2(x) = \sum_{1 \leq m_1 \leq m_2 \leq x} Q(x/m_2, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}). \tag{5.1}$$

We next split the summation so that

$$F_2(x) = \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} + \sum_{1 \leq m_1 \leq m_2 \leq x;\, m_2 > x^{1/3}} Q\left(\frac{x}{m_2}, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}\right)$$
$$= S_1 + S_2.$$

We estimate S_1 first. Applying Lemma 3.2 we have

$$S_1 = \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} \frac{x}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}} \sum_d \frac{\mu(d)}{d^2} (d, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})$$
$$+ O\left(\sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} 2^{\nu(\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})} \left(\frac{x}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}}\right)^{1/2}\right).$$

We note first that since $2^{\nu(t)} = O(t^{\epsilon'})$ the O term is less than

$$x^{1/2+\epsilon'} \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} \frac{1}{(m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})^{1/2}} \leqq x^{1/2+\epsilon'} \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} \frac{m_1^{1/2}}{\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}}$$
$$\leqq x^{2/3+\epsilon'} \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}^{-1}$$

and using (3.7) this is $O(x^{2/3+\epsilon})$. Thus

$$S_1 = x \sum_d \frac{\mu(d)}{d^2} \sum_{1 \leq m_1 \leq m_2 \leq x^{1/3}} \frac{(d, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}} + O(x^{2/3+\epsilon}). \tag{5.2}$$

Applying Lemma 4.1 with $U = x^{1/3}$, $V = \infty$, yields that

$$x \sum_d \frac{\mu(d)}{d^2} \sum_{1 \leq m_1 \leq m_2;\, m_2 > x^{1/3}} \frac{(d, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}} = O(x^{2/3+\epsilon}). \tag{5.3}$$

Thus (5.2) becomes

$$S_1 = cx + O(x^{2/3+\epsilon}) \tag{5.4}$$

where c is a constant given by

$$c = \sum_d \frac{\mu(d)}{d^2} \sum_{1 \leq m_1 \leq m_2} \frac{(d, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})}{m_2\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}}.$$

In fact it is the estimate (5.3) which establishes the convergence of the series.

Turning next to S_2, we have

$$S_2 \leqq \sum_{1 \leq m_1 \leq m_2 \leq x} Q(x^{2/3}, \{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\})$$
$$\leqq x^{2/3} \sum_{1 \leq m_1 \leq m_2 \leq x} \frac{1}{\{\mathrm{sq}(m_1), \mathrm{sq}(m_2)\}}$$

and from (3.7), this is $O(x^{2/3+\epsilon})$.

Since $F_2(x) = S_1 + S_2$, we have that

$$F_2(x) = cx + O(x^{2/3+\epsilon})$$

as asserted in (1.1).

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