Rate of Convergence Estimates for Nonselfadjoint Eigenvalue Approximations*

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Abstract. In this paper, a general approximation theory for the eigenvalues and corresponding subspaces of generalized eigenfunctions of a certain class of compact operators is developed. This theory is then used to obtain rate of convergence estimates for the errors which arise when the eigenvalues of nonselfadjoint elliptic partial differential operators are approximated by Rayleigh-Ritz-Galerkin type methods using finite-dimensional spaces of trial functions, e.g. spline functions. The approximation methods include several in which the functions in the space of trial functions are not required to satisfy any boundary conditions.

1. Introduction. In this paper, we develop a general approximation theory for the eigenvalues and corresponding subspaces of generalized eigenvectors for a certain class of compact operators. This theory is then applied to several examples. The principal examples consist of certain Rayleigh-Ritz-Galerkin type methods for the approximation of the eigenvalues of second-order nonselfadjoint elliptic differential operators. In each case we give estimates for the rate of convergence. By means of our general theorems on compact operators, the problem of estimating the rate of convergence of the eigenvalues is reduced to obtaining estimates for the rate of convergence in the approximate solution of the corresponding boundary value problem. We make strong use of not only $\mathcal{L}_2$ estimates but also estimates in Sobolev spaces of negative order.

Although our theorems could be applied to higher-order elliptic equations, ordinary differential equations and integral operators, we have chosen to center our attention on the general second-order elliptic case and to obtain specific results for a variety of projection methods.

There is an extensive literature dealing with the approximation of eigenvalues of selfadjoint operators. For the case of a selfadjoint operator which is bounded below, upper bounds for the eigenvalues are provided by the Rayleigh-Ritz method. The method of intermediate problems introduced by Weinstein [47], [48] provides lower bounds. This method was extended by Weinstein [49], [50], [51], [52], Aronszajn and Weinstein [3], Aronszajn [1], [2], Weinberger [45], [46], Bazley [7], Bazley and Fox [8], [9], and Stenger [41]. Another method has been developed by Fichera [21], [22] for finding lower bounds for eigenvalues connected with selfadjoint boundary value problems for elliptic differential operators in a bounded domain.

Also, in the selfadjoint case, Birkhoff, de Boor, Swartz, and Wendroff [11] prove an inequality from which lower bounds can be derived for the eigenvalues in terms

of the Rayleigh-Ritz eigenvalues corresponding to a given finite-dimensional subspace and the approximability properties of the subspace. A specific application of this was carried out for second-order ordinary differential operators using subspaces of cubic splines. This idea has been generalized so as to apply to selfadjoint elliptic operators and rather general finite-dimensional subspaces; cf. Ciarlet, Schultz, and Varga [19], Schultz [39], [40]. Pierce and Varga [33], [34] give $\mathcal{L}_2$ eigenvector estimates which are optimal in the sense of corresponding to the approximability properties of the underlying subspaces.

Perhaps the work closest to that of this paper is the work of Vaĭnikko [42], [43], [44]. In [44], he proves a general theorem on convergence of eigenvalues and generalized eigenvectors. Our Theorems 3 and 4 of Section 3 are analogous to his. He does not obtain results of the type given in Theorems 1 and 2 of Section 3 (our main theorems) and it is not clear how to put several of our applications into his setting. His strongest results rest on the fact that an operator T is approximated by an operator $P_h T$ where P_h is a certain projection whose adjoint can be calculated. This task, which must be carried out with each application, does not appear to be simple. Our approach, while avoiding this difficulty, yields stronger results.

Recently, Osborn [30], [31], [32] has derived error estimates for Galerkin approximations of the eigenvalues of nonselfadjoint ordinary differential operators which are lower order perturbations of selfadjoint operators. The eigenfunctions of the related selfadjoint operator are used as trial functions and the error estimates depend in a simple way on the data of the problem.

We mention finally that Marek [27] gives error estimates for the approximation of the Perron root of a class of nonselfadjoint operators which are of positive type provided the approximate operator is also of positive type.

Error estimates for the approximate calculation of eigenvalues of selfadjoint operators have always depended on a variational characterization of the eigenvalues. In the present work, no use is made of such variational principles; instead, we use certain estimates for the related resolvent operator. Special features of this work are:

1. Approximations for the generalized eigenfunctions are obtained first and then used to obtain the eigenvalue estimates.

2. In the case of a multiple eigenvalue (in general having different algebraic and geometric multiplicities), a weighted average of approximate eigenvalues is shown to be the "right" choice as an approximation.

3. The results apply to general nonselfadjoint elliptic operators.

4. Various methods for defining the approximate eigenvalues are treated, including methods in which the functions in the subspaces of trial functions are not required to satisfy any boundary conditions.

5. The estimates are optimal in the sense that they give the best estimate that could be expected on the basis of the approximability properties of the underlying subspaces.

An outline of the paper is as follows: In Sections 2 and 3, we derive results on the approximation of eigenvalues and generalized eigenfunctions of a class of compact operators. Section 4 deals with the general class of differential eigenvalue problems to be considered in the paper. In Section 5 we discuss the families of finite-dimensional subspaces used in constructing the approximations. Sections 6–11 contain various smethods for constructing the approximate eigenvalues: The ordinary Galerkin

method, the least squares method of Bramble and Schatz [15], [16], methods of Nitsche [28], [29], and a recent method of Babuška [6]. In each case the eigenvalue results are shown to follow easily from results on the approximation of the solution of the corresponding inhomogeneous problem.

2. General Notation. Let Ω be a bounded domain in Euclidean N-dimensional space R^N with boundary $\partial\Omega$ which we will assume (for convenience) to be of class C^∞. Let $C^\infty(\bar{\Omega})$ be the class of infinitely differentiable complex valued functions defined on $\bar{\Omega}$, the closure of Ω. For functions in $C^\infty(\bar{\Omega})$, we define the $\mathcal{L}_2$-inner product by

$$(\varphi, \psi) = \int_\Omega \varphi \bar{\psi} \, dx$$

and the corresponding norm by

$$\|\varphi\|_0 = \left(\int_\Omega |\varphi|^2 \, dx \right)^{1/2}.$$

Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ be a multi-index (i.e., each α_i, $i = 1, \cdots, N$, is a nonnegative integer),

$$D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_N}/\partial x_N^{\alpha_N}$$

and $|\alpha| = \sum_{i=1}^N \alpha_i$. We define for $\varphi \in C^\infty(\bar{\Omega})$ and for j a nonnegative integer the norm

$$\|\varphi\|_j = \left(\sum_{|\alpha| \leq j} \|D^\alpha \varphi\|_0^2 \right)^{1/2}.$$

The completion of $C^\infty(\bar{\Omega})$ with respect to $\|\cdot\|_j$ will be denoted by $H^j(\Omega)$. This is the Sobolev space of order j and is a Hilbert space. For any positive real number s, we define $H^s(\Omega)$ by interpolation between successive integers following Definition 2.1 of [26]. The spaces $H^s(\Omega)$ for $s < 0$ are defined as follows:

For $\varphi \in C^\infty(\bar{\Omega})$ and $s < 0$, set

(2.1) $$\|\varphi\|_s = \sup_{\psi \in C^\infty(\bar{\Omega})} |(\varphi, \psi)|/\|\psi\|_{-s}.$$

Then $H^s(\Omega)$, for $s < 0$, is defined as the completion of $C^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_s$ (cf. [10]). Now $H^s(\Omega)$ is a Hilbert space for any real s. Note that $H^0(\Omega) = \mathcal{L}_2(\Omega)$ and that $H^{s_2}(\Omega) \subset H^{s_1}(\Omega)$ if $s_1 \leq s_2$. We state the following well-known result.

LEMMA 2.1 (RELLICH). *Let s_1 and s_2 be real numbers with $s_1 < s_2$, and let I_{s_1,s_2} be the injection operator mapping $H^{s_2}(\Omega)$ into $H^{s_1}(\Omega)$. Then I_{s_1,s_2} is compact.*

For any $A: H^s(\Omega) \to H^s(\Omega)$, we define, for $s_1 \leq s \leq s_2$,

$$\|A\|_{s_1,s_2} = \sup_{\varphi \in H^{s_2}(\Omega)} \|A\varphi\|_{s_1}/\|\varphi\|_{s_2}.$$

Note that, for each given s_1, s_2 and s, there is a constant C such that

(2.2) $$\|A\|_{s_1,s_2} \leq C \|A\|_{s,s}.$$

The restriction of A to a subset X of $H^s(\Omega)$ will be denoted by $A|_X$. Let $\sigma(A)$ denote the spectrum of A and $\rho(A)$ the resolvent set. For any complex number $z \in \rho(A)$, the resolvent operator $R_z(A)$ is defined by

$$R_z(A) = (z - A)^{-1}.$$

If A is compact, then $\sigma(A)$ consists of a denumerable set of nonzero complex numbers and zero. Each nonzero point μ of $\sigma(A)$ is an eigenvalue of A; i.e., there exists a nonzero element u of $H^s(\Omega)$ such that $Au = \mu u$. Zero may or may not be an eigenvalue.

Let μ be a nonzero eigenvalue of A. The least integer α such that $\mathfrak{N}((A - \mu)^\alpha) = \mathfrak{N}((A - \mu)^{\alpha+1})$, where $\mathfrak{N}$ denotes the null space, is called the ascent of $A - \mu$. If A is compact α is finite. The integer $m = \dim \mathfrak{N}((A - \mu)^\alpha)$ is called the algebraic multiplicity of μ and is likewise finite. The subspace $\mathfrak{N}((A - \mu)^\alpha)$ is called the space of generalized eigenfunctions corresponding to the eigenvalue μ. The geometric multiplicity is equal to $\dim \mathfrak{N}(A - \mu)$ and is always less than or equal to m. In the case of a selfadjoint operator, the geometric and algebraic multiplicities of a given eigenvalue μ are equal.

We next state a general result on the approximation of eigenvalues of compact operators.

Let $K: \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$ be compact. Suppose that $\{K_h\}_{0 < h \leq 1}$ is a family of compact operators $K_h: \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$ such that

$$\lim_{h \to 0} \| K - K_h \|_{0,0} = 0.$$

Let $\mu_1, \mu_2, \cdots$ be the nonzero eigenvalues of K ordered by decreasing magnitude taking account of algebraic multiplicities. Then for each $h > 0$ there is an ordering of the eigenvalues of K_h, $\mu_1(h), \mu_2(h), \cdots$, such that, for each integer j, $\lim_{h \to 0} \mu_j(h) = \mu_j$ (cf. [20]).

For the purpose of this paper, we shall restate this convergence property in the following convenient form.

LEMMA 2.2. *Let μ be a nonzero eigenvalue of K with algebraic multiplicity m and let Γ be a circle centered at μ which lies in $\rho(K)$ and contains no other points of $\sigma(K)$. Then, there is an h_0 such that, for $0 < h \leq h_0$, there are exactly m eigenvalues (counting algebraic multiplicities) of K_h lying inside Γ and all points of $\sigma(K_h)$ are bounded away from Γ.*

The next lemma relates the projection onto the generalized null space corresponding to μ to the projection associated with the part of the spectrum of K_h inside Γ (cf. [25]).

LEMMA 2.3. *With Γ and h_0 as in Lemma 2.2, if $0 < h \leq h_0$, the operators*

$$E(\mu) = \frac{1}{2\pi i} \int_\Gamma R_z(K)\, dz \quad and \quad E_h(\mu) = \frac{1}{2\pi i} \int_\Gamma R_z(K_h)\, dz$$

are projections (in general nonorthogonal). The range of $E(\mu)$, $R(E(\mu))$, is the space of generalized eigenfunctions of K corresponding to μ and $R(E_h(\mu))$ is the direct sum of the spaces of generalized eigenfunctions of K_h associated with the eigenvalues of K_h inside of Γ. Finally,

$$\lim_{h \to 0} ||E(\mu) - E_h(\mu)||_{0,0} = 0$$

and dim $R(E_h(\mu)) = $ dim $R(E(\mu)) = m$.

3. Convergence Estimates. In this section we will consider a particular type of compact operator T and a family of compact operators $\{T_h\}_{0 < h \leq 1}$ which approximates it. The first theorem relates a certain invariant subspace of T_h to generalized eigenfunctions of T. These estimates are then used to deduce estimates for the rate of convergence of the eigenvalues of T_h to the eigenvalues of T.

More precisely let s_0 be a fixed nonnegative real number. Let T be an operator from $H^{-s_0}(\Omega)$ to $H^{-s_0}(\Omega)$. We suppose further that for some fixed $\epsilon > 0$ and all $s \geq -s_0$,

(3.1) $$T: H^s(\Omega) \to H^{s+\epsilon}(\Omega)$$

as a bounded mapping.

From (3.1) we see that T can be considered as an operator from $H^s(\Omega)$ to $H^s(\Omega)$ for $s \geq -s_0$ and, from Lemma 2.1 we see that T will be compact as an operator on $H^s(\Omega)$. In the sequel the space $H^s(\Omega)$ on which we are considering T will be clear from the context.

LEMMA 3.1. *For $s \geq -s_0$, the spectrum of T on $H^s(\Omega)$ is the same as the spectrum of T on $\mathcal{L}_2(\Omega)$.*

Proof. Clearly, the spectrum of T on $H^{-s_0}(\Omega)$ contains the spectrum of T on $H^s(\Omega)$. Suppose $\mu \neq 0$ is in the spectrum of T on $H^{-s_0}(\Omega)$. Then, there is a nonzero $u \in H^{-s_0}(\Omega)$ such that $Tu = \mu u$. By virtue of (3.1), $u \in H^t(\Omega)$ for all $t \geq -s_0$. Hence, $u \in H^s(\Omega)$ and it follows that μ is in the spectrum of T on $H^s(\Omega)$.

Note that the space of generalized eigenfunctions corresponding to a nonzero eigenvalue is also independent of s. This follows easily from (3.1).

Now let $\{T_h\}_{0 < h \leq 1}$ be a one-parameter family of compact operators from $L_2(r)$ to $L_2(r)$ such that

(3.2) $$\lim_{h \to 0} ||T - T_h||_{0,0} = 0.$$

Let μ be a nonzero eigenvalue of T with algebraic multiplicity m. Let Γ be a circle centered at μ which lies in $\rho(T)$ and contains no other points of $\sigma(T)$. From Lemma 2.2 applied to T and $\{T_h\}_{0 < h \leq 1}$ we see that there is an h_0 such that, for $0 < h \leq h_0$ there are exactly m eigenvalues (counting algebraic multiplicities) of T_h inside of Γ. Throughout this section, μ, Γ, and h_0 will be considered fixed and the m eigenvalues of T_h which lie inside Γ are denoted by $\mu_1(h), \cdots, \mu_m(h)$.

Now, from Lemma 2.3, with $K = T|_{\mathcal{L}_2(\Omega)}$, $K_h = T_h$ and $h \leq h_0$, the operators

$$E(\mu) = \frac{1}{2\pi i} \int_\Gamma R_z(T|_{\mathcal{L}_2(\Omega)}) \, dz \quad \text{and} \quad E_h(\mu) = \frac{1}{2\pi i} \int_\Gamma R_z(T_h) \, dz$$

are projections onto the space of generalized eigenfunctions of T corresponding to μ and the direct sum of the spaces of generalized eigenfunctions of T_h corresponding to $\mu_1(h), \cdots, \mu_m(h)$, respectively, and dim $R(E_h(\mu)) = $ dim $R(E(\mu)) = m$.

The following lemma is central to our development.

LEMMA 3.2. *Let s_1 be a given nonnegative real number. Then there are constants C and h_1 such that for any $u \in H^{s_1}(\Omega)$, $0 < h \leq h_1$ and $0 \leq s \leq s_0$,*

(3.3) $||E(\mu)u - E_h(\mu)u||_{-s}$

$$\leqq C\{||T - T_h||_{-s,s_1} + ||T - T_h||_{-s,0} \, ||T - T_h||_{0,s_1}\} \cdot ||u||_{s_1}.$$

Proof. By (2.1), for $0 \leqq s \leqq s_0$,

(3.4) $$||E(\mu)u - E_h(\mu)u||_{-s} = \sup_{\varphi \in C^\infty(\bar{\Omega})} |([E(\mu) - E_h(\mu)]u, \varphi)|/||\varphi||_s$$

$$= \frac{1}{2\pi} \sup_{\varphi \in C^\infty(\bar{\Omega})} \left| \int_\Gamma \frac{([R_z(T) - R_z(T_h)]u, \varphi)}{||\varphi||_s} \, dz \right|.$$

Now let $z \in \Gamma$ be arbitrary. We have for any $\varphi \in H^s(\Omega)$,

(3.5) $$|([R_z(T) - R_z(T_h)]u, \varphi)|/||\varphi||_s \leqq ||[R_z(T) - R_z(T_h)]u||_{-s}.$$

Consider now

(3.6) $$[R_z(T) - R_z(T_h)]u = R_z(T)(T - T_h)R_z(T)u$$

$$- R_z(T)(T - T_h)R_z(T_h)(T - T_h)R_z(T)u.$$

It follows easily from (3.6) that

(3.7) $$||[R_z(T) - R_z(T_h)]u||_{-s}$$

$$\leqq ||R_z(T)||_{-s,-s} \, ||T - T_h||_{-s,s_1} \, ||R_z(T)||_{s_1,s_1} \, ||u||_{s_1}$$

$$+ ||R_z(T)||_{-s,-s} \, ||T - T_h||_{-s,0} \, ||R_z(T_h)||_{0,0}$$

$$\cdot ||T - T_h||_{0,s_1} \, ||R_z(T)||_{s_1,s_1} \, ||u||_{s_1}.$$

Now by Lemma 3.1, z is in the resolvent set of T on $H^t(\Omega)$ for $-s_0 \leqq t$. Since the resolvent of T on $H^t(\Omega)$ is continuous as a function of z and Γ is compact, we have for some constant C that

(3.8) $$||R_z(T)||_{t,t} \leqq C$$

for all $z \in \Gamma$ and $-s_0 \leqq t \leqq s_1$.

Furthermore,

$$R_z(T_h) = (I + R_z(T)(T - T_h))^{-1}R_z(T)$$

provided $||R_z(T)(T - T_h)||_{0,0} < 1$. By (3.3) and (3.8), h_1 may be chosen so that this condition is fulfilled uniformly for $0 < h \leqq h_1$ and $z \in \Gamma$. Thus there is a constant C such that for $0 < h \leqq h_1$ and all $z \in \Gamma$,

(3.9) $$||R_z(T_h)||_{0,0} \leqq C.$$

Using (3.9), (3.8), and (3.7), we have

(3.10) $||[R_z(T) - R_z(T_h)]u||_{-s}$

$$\leqq C\{||T - T_h||_{-s,s_1} + ||T - T_h||_{-s,0} \, ||T - T_h||_{0,s_1}\} \, ||u||_{s_1}.$$

Inequality (3.4) now follows from (3.10), (3.5) and (3.4).

The next lemma is an easy consequence of Lemma 3.2. In what follows, the term orthonormal will always refer to $\mathcal{L}_2(\Omega)$.

LEMMA 3.3.   *Let $\{u_j\}_{j=1}^m$ be an orthonormal basis for $R(E(\mu))$ and $s_1 \geqq 0$. There exist constants C_1 and h_2 and a basis $\{\tilde{w}_j\}_{j=1}^m$ for $R(E_h(\mu))$ such that for $0 \leqq s \leqq s_0$ and $0 < h \leqq h_2$,*

$$(3.11) \quad \max_{1 \leqq i \leqq m} ||u_j - \tilde{w}_j||_{-s} \leqq C_1\{||T - T_h||_{-s,s_1} + ||T - T_h||_{-s,0}\,||T - T_h||_{0,s_1}\}.$$

Proof.   Since $u_j \in R(E(\mu))$ and $E(\mu)$ is a projection, we have that $u_j = E(\mu)u_j$. Since $u_j \in H^{s_1}(\Omega)$ we may apply Lemma 3.2 with $u = u_j$, $j = 1, \cdots, m$. Hence, choosing $C_1 = C \max_j ||u_j||_{s_1}$ with C the constant from Lemma 3.2, (3.11) follows with $\tilde{w}_j = E_h(\mu)u_j$. Now by (2.2),

$$\left.\begin{array}{c} ||T - T_h||_{-s,s_1} \\[2mm] ||T - T_h||_{-s,0} \\[2mm] ||T - T_h||_{0,s_1} \end{array}\right\} \leqq C'||T - T_h||_{0,0}$$

and hence, because of (3.2) there is a constant h_2 such that for $0 < h \leqq h_2$,

$$\sum_{j=1}^m ||u_j - \tilde{w}_j||_0^2 < 1.$$

But from this it follows that the $\tilde{w}_j$'s are linearly independent and hence a basis for the m-dimensional space $R(E_h(\mu))$.

The first of our principal aims of this section is to obtain an estimate like (3.11) but for an orthonormal basis for $R(E_h(\mu))$. To this end we prove the following general lemma. Because of our application we phrase this lemma in $\mathscr{L}_2(\Omega)$ even though it is true with $\mathscr{L}_2(\Omega)$ replaced by any inner product space.

LEMMA 3.4.   *Let m be a positive integer. There is a constant C_m such that, for $f_1, \cdots, f_m$ any linearly independent set in $\mathscr{L}_2(\Omega)$ and $g_1, \cdots, g_m$ the corresponding Gram-Schmidt orthonormalization, we have*

$$(3.12) \quad \max_{1 \leqq k \leqq m} ||f_k - g_k||_0 \leqq C_m \max_{1 \leqq j, k \leqq m} |(f_j, f_k) - \delta_{jk}|,$$

where δ_{jk} is the Kronecker delta.

Proof.   We proceed by induction on m. Let $M = \max_{1 \leqq j, k \leqq m} |(f_j, f_k) - \delta_{jk}|$. Observe that

$$(3.13) \quad |\,||f_m||_0 - 1| \leqq |\,||f_m||_0^2 - 1| \leqq M.$$

In particular, for $m = 1$, we have $||f_1 - g_1||_0 = |\,||f_1||_0 - 1| \leqq M$. Now assume the result holds for $m - 1$ with $m \geqq 2$ and a constant C_{m-1}. Set

$$(3.14) \quad \tilde{g}_m = f_m - \sum_{j=1}^{m-1} (f_m, g_j)g_j$$

and consider first the case $M \leqq 1$. Since

$$|(f_m, g_j)| \leqq ||f_m||_0\,||g_j - f_j||_0 + |(f_m, f_j)|$$

it follows from the induction hypothesis and (3.13) that

$$|(f_m, g_j)| \leqq (2C_{m-1} + 1)M$$

for $j < m$ and hence, from (3.14),

(3.15) $||f_m - \tilde{g}_m||_0 \leqq (m - 1)^{1/2}(2C_{m-1} + 1)M.$

Now, since $g_m = \tilde{g}_m/||\tilde{g}_m||_0,$

$$||\tilde{g}_m - g_m||_0 = |||\tilde{g}_m||_0 - 1|$$

$$\leqq |||\tilde{g}_m||_0 - ||f_m||_0| + |||f_m||_0 - 1|$$

(3.16)

$$\leqq ||\tilde{g}_m - f_m||_0 + |||f_m||_0 - 1|$$

$$\leqq [(m - 1)^{1/2}(2C_{m-1} + 1) + 1]M.$$

For $M \leqq 1$ the result follows from (3.15), (3.16) and the triangle inequality. If $M > 1$ we have

$$||f_m - g_m||_0 \leqq ||f_m||_0 + 1 \leqq 2 + M \leqq 3M$$

and hence the lemma is proved.

We shall also need:

LEMMA 3.5. *Let* $s \geqq 0$ *be given and let* $\{u_j\}_{j=1}^m$ *be an orthonormal set in* $\mathfrak{L}_2(\Omega)$ *such that* $u_j \in H^s(\Omega)$ *for each* j. *There exists a constant* C *such that if* $\{\tilde{w}_j\}_{j=1}^m$ *is any linearly independent set and* $\{w_j\}_{j=1}^m$ *is its Gram-Schmidt orthonormalization then*

$$\max_{1 \leqq k \leqq m} ||w_k - \tilde{w}_k||_0 \leqq C \max_{1 \leqq j \leqq m} (||u_j - \tilde{w}_j||_{-s} + ||u_j - \tilde{w}_j||_0^2).$$

Proof. We have, from Lemma 3.4, a constant C depending only on m such that

(3.17) $\max\limits_{1 \leqq k \leqq m} ||w_k - \tilde{w}_k||_0 \leqq C \max\limits_{1 \leqq i,j \leqq m} |(\tilde{w}_i, \tilde{w}_j) - \delta_{ij}|.$

Now, for fixed i and j,

$$|(\tilde{w}_i, \tilde{w}_j) - \delta_{ij}| = |(\tilde{w}_i, \tilde{w}_j) - (u_i, u_j)|$$

(3.18)

$$= |(\tilde{w}_i - u_i, \tilde{w}_j - u_j) + (u_i, \tilde{w}_j - u_j) + (\tilde{w}_i - u_i, u_j)|$$

$$\leqq ||u_i - \tilde{w}_i||_0 \, ||u_j - \tilde{w}_j||_0 + ||u_i||_s \, ||u_j - \tilde{w}_j||_{-s}$$

$$+ ||u_j||_s \, ||u_i - \tilde{w}_i||_{-s}.$$

The result follows easily from (3.17) and (3.18).

We now state and prove the first principal result of this section.

THEOREM 1. *Let* $\{u_j\}_{j=1}^m$ *be an orthonormal basis for* $R(E(\mu))$ *and* $s_1 \geqq 0$. *There exist constants* C *and* h_2 *and an orthonormal basis* $\{w_j\}_{j=1}^m$ *for* $R(E_h(\mu))$ *such that, for* $0 \leqq s \leqq s_0$ *and* $0 < h \leqq h_2$,

(3.19) $\max\limits_{1 \leqq k \leqq m} ||u_k - w_k||_{-s}$

$$\leqq C\{||T - T_h||_{-s,s_1} + ||T - T_h||_{-s,0} \, ||T - T_h||_{0,s_1} + ||T - T_h||_{0,s_1}^2\}.$$

Proof. Let $\{\tilde{w}_j\}_{j=1}^m$ be the basis for $R(E_h(\mu))$ given by Lemma 3.3 and $\{w_j\}_{j=1}^m$ its Gram-Schmidt orthonormalization. Then

$$||u_k - w_k||_{-s} \leqq ||u_k - \tilde{w}_k||_{-s} + ||\tilde{w}_k - w_k||_0.$$

Applying Lemma 3.5 we obtain, for an appropriate constant C,

(3.20) $||w_k - \tilde{w}_k||_0 \leqq C \max\limits_{1 \leqq j \leqq m} (||u_j - \tilde{w}_j||_{-s} + ||u_j - \tilde{w}_j||_0^2).$

The theorem now follows by using the estimate (3.11) on the right-hand side of (3.20).

We may also start with an orthonormal basis in $R(E_h(\mu))$ and obtain a close orthonormal basis for $R(E(\mu))$. This we see in the following:

COROLLARY 1. *For each h with $0 < h \leq h_2$ let $\{w_j\}_{j=1}^m$ be an orthonormal basis for $R(E_h(\mu))$. Then there is an orthonormal basis $\{u_j\}_{j=1}^m$ for $R(E(\mu))$ such that*

$$\max_{1 \leq i \leq m} \|u_i - w_i\|_{-s}$$

$$\leq C\{\|T - T_h\|_{-s,s_1} + \|T - T_h\|_{-s,0} \|T - T_h\|_{0,s_1} + \|T - T_h\|_{0,s_1}^2\}$$

holds for all $0 \leq s \leq s_0$, where C is a constant independent of h.

Proof. Let $\{u'_j\}_{j=1}^m$ be an orthonormal basis for $R(E(\mu))$ and let $\{w'_j\}_{j=1}^m$ be the corresponding orthonormal basis in $R(E_h(\mu))$ given by Theorem 1. Now $w_j = \sum_{k=1}^m (w_j, w'_k)w'_k$. Set $u_j = \sum_{k=1}^m (w_j, w'_k)u'_k$. The result follows by applying the estimate of Theorem 1 to each term on the right-hand side of the expression $u_j - w_j = \sum_{k=1}^m (w_j, w'_k)(u'_k - w'_k)$.

Theorem 1 and Corollary 1 show that the "gap" between $R(E_h(\mu))$ and $R(E(\mu))$ is estimated by the right-hand side of (3.19) (cf. [25]).

We want next to study the eigenvalues of T_h which are inside Γ and their relationship to μ. As before, let $\mu_1(h), \cdots, \mu_m(h)$ denote the eigenvalues of T_h which lie inside Γ. In case they are not all distinct, they are counted according to their algebraic multiplicities. Now, it is well known that the individual $\mu_j(h)$'s may be rather poor approximations to μ because of the nonselfadjointness of T although by Lemma 2.2 they converge to μ as $h \to 0$. However, their arithmetic mean is a much better approximation to μ (cf. [25]). We therefore define

$$\hat{\mu}(h) = \frac{1}{m} \sum_{j=1}^m \mu_j(h).$$

The next lemma gives representations for μ and $\hat{\mu}(h)$ in terms of T, T_h and elements of the subspaces $R(E(\mu))$ and $R(E_h(\mu))$.

LEMMA 3.6. *Let $\{u_j\}_{j=1}^m$ be any orthonormal basis for $R(E(\mu))$ and $\{w_j\}_{j=1}^m$ be any orthonormal basis for $R(E_h(\mu))$ with $0 < h \leq h_0$. Then*

$$(3.21) \qquad \mu = \frac{1}{m} \sum_{j=1}^m (Tu_j, u_j)$$

and

$$(3.22) \qquad \hat{\mu}(h) = \frac{1}{m} \sum_{j=1}^m (T_h w_j, w_j).$$

Proof. Denote by $\tilde{T}$ the restriction of T to $R(E(\mu))$ and by $\tilde{T}_h$ the restriction of T_h to $R(E_h(\mu))$. We first observe, since $R(E(\mu))$ is an invariant subspace for T, that $\sigma(\tilde{T}) = \{\mu\}$ and that μ is an eigenvalue of $\tilde{T}$ with algebraic multiplicity m. Similarly, for $0 < h \leq h_0$, $\sigma(\tilde{T}_h) = \{\mu_1(h), \cdots, \mu_m(h)\}$. Since on the one hand trace $\tilde{T} = m\mu$ and on the other hand trace $\tilde{T}$ is equal to the sum of the diagonal elements of any one of its matrix representations, (3.21) follows. In the same way we obtain (3.22).

We are now in a position to state and prove the second main result of this section.

THEOREM 2. *Let s_1 be a given nonnegative real number. Then there exist constants C and h_2 such that, for $0 < h \leq h_2$,*

(3.23) $\quad |\mu - \hat{\mu}(h)|$

$$\leqq C\{||T - T_h||_{-s_0, s_1} + ||T - T_h||_{-s_0, 0}\, ||T - T_h||_{0, s_1} + ||T - T_h||^2_{0, s_1}\}.$$

Proof. Let $\{u_j\}^m_{j=1}$ be any orthonormal basis for $R(E(\mu))$ and let $\{w_j\}^m_{j=1}$ be the corresponding orthonormal basis for $R(E_h(\mu))$ given by Theorem 1. Then, by Lemma 3.6, we have

$$(3.24) \qquad\qquad \mu - \hat{\mu}(h) = \frac{1}{m} \sum_{j=1}^{m} [(Tu_j, u_j) - (T_h w_j, w_j)].$$

We will obtain the estimate (3.23) by re-expressing the right-hand side of (3.24).

For ease of notation let j be fixed with $1 \leqq j \leqq m$ and set $u_j = u$ and $w_j = w$. Then

$$\begin{aligned}(Tu, u) - (T_h w, w) &= (T(u - w), u) - ((T - T_h)(u - w), u) \\ (3.25) &\qquad + (Tu, u - w) - ((T - T_h)u, u - w) \\ &\qquad + ((T - T_h)u, u) - (T_h(u - w), u - w).\end{aligned}$$

We estimate the terms on the right-hand side of (3.25) as follows:

$$|(T(u - w), u)| \leqq ||T||_{-s_0, -s_0}\, ||u - w||_{-s_0}\, ||u||_{s_0},$$

$$|((T - T_h)(u - w), u)| \leqq ||T - T_h||_{-s_0, 0}\, ||u - w||_0\, ||u||_{s_0},$$

$$|(Tu, u - w)| \leqq ||T||_{s_0, s_0}\, ||u||_{s_0}\, ||u - w||_{-s_0},$$

$$|((T - T_h)u, u - w)| \leqq ||T - T_h||_{0, s_1}\, ||u||_{s_1}\, ||u - w||_0,$$

$$|((T - T_h)u, u)| \leqq ||T - T_h||_{-s_0, s_1}\, ||u||_{s_1}\, ||u||_{s_0},$$

$$|(T_h(u - w), u - w)| \leqq ||T_h||_{0, 0}\, ||u - w||^2_0.$$

Since $||T - T_h||_{0,0} \to 0$ as $h \to 0$, $||T_h||_{0,0}$ is bounded uniformly in h for $0 < h \leqq h_2$. Now each expression on the right-hand side of each of the six inequalities is bounded by a constant times the right-hand side of (3.19) with $s = s_0$. Hence, using these estimates with (3.25) and (3.24), the proof of Theorem 2 is complete.

We may also estimate $\mu - \mu_k(h)$ for each k.

THEOREM 3. *Let α be the ascent of $\mu - T$. Then there are constants C and h_2 such that, for $0 < h \leqq h_2$ and $s_1 \geqq 0$,*

$$\max_{1 \leqq k \leqq m} |\mu - \mu_k(h)|^\alpha$$

$$\leqq C\{||T - T_h||_{-s_0, s_1} + ||T - T_h||_{-s_0, 0}\, ||T - T_h||_{0, s_1} + ||T - T_h||^2_{0, s_1}\}.$$

Proof. Let w be a unit eigenfunction corresponding to $\mu_k(h)$. By Corollary 1 there is a unit vector $u \in R(E(\mu))$ with $||u - w||_{-s}$ estimated by the right-hand side of (3.19). Now, noting that $(\mu - T)^\alpha u = 0$, we have

$$\begin{aligned}|\mu - \mu_k(h)|^\alpha &= |([(\mu - \mu_k(h))^\alpha - (\mu - T)^\alpha]u, u)| \\ &= \left| \sum_{j=0}^{\alpha-1} (\mu - \mu_k(h))^j ((\mu - T)^{\alpha-1-j}(\mu_k(h) - T)u, u) \right|.\end{aligned}$$

Since $R(E(\mu))$ consists of generalized eigenfunctions, $||u||_s$ will be bounded in h for

any fixed s even though u may depend on h. Using this and the fact that T is bounded in $H^{-s}{}^{\circ}(\Omega)$ we obtain

$$|\mu - \mu_k(h)|^\alpha \leqq C \, ||(\mu_k(h) - T)u||_{-s_0}.$$

Adding and subtracting w and observing that $(\mu_k(h) - T_h)w = 0$ gives

$$|\mu - \mu_k(h)|^\alpha$$

$$\leqq C\{||(\mu_k(h) - T)(u - w)||_{-s_0} + ||(T - T_h)(u - w)||_{-s_0} + ||(T - T_h)u||_{-s_0}\}$$

from which we obtain

$$|\mu - \mu_k(h)|^\alpha \leqq C\{||u - w||_{-s_0} + ||T - T_h||_{-s_0,0} \, ||u - w||_0 + ||T - T_h||_{-s_0,s_1}\}.$$

The result follows immediately from Corollary 1.

Finally in this section we obtain quite easily results concerning the proximity of certain elements of $R(E_h(\mu))$ to certain elements of $R(E(\mu))$. For example, for small h an eigenvector in $R(E_h(\mu))$ will be close to an eigenvector in $R(E(\mu))$. The full result is given in the following:

THEOREM 4. *Let $\mu(h)$ be an eigenvalue of T_h such that $\mu(h) \to \mu$ as $h \to 0$ and $s_1 \geqq 0$. Suppose, for each h, w is a unit vector satisfying $(\mu(h) - T_h)^k w = 0$ for some positive integer $k \leqq \alpha$. Then, for any integer l with $k \leqq l \leqq \alpha$, there is a vector $u \in R(E(\mu))$ such that $(\mu - T)^l u = 0$ and*

$$||u - w||_0 \leqq C \, ||T - T_h||_{0,s_1}^{(l-k+1)/\alpha}.$$

Proof. From the Riesz-Schauder theory and the closed graph theorem it follows that $(\mu - T|_{\mathcal{L}_2(\Omega)})^l$ has a bounded right inverse which maps $R((\mu - T|_{\mathcal{L}_2(\Omega)})^l)$ to the orthogonal complement of $\mathfrak{N}(\mu - T)^l$ in $\mathcal{L}_2(\Omega)$. Thus, setting $u = Pw$ where P is the orthogonal projection of $\mathcal{L}_2(\Omega)$ onto $\mathfrak{N}((\mu - T)^l)$, there is a constant C independent of w such that

(3.26) $$||w - u||_0 \leqq C \, ||(\mu - T)^l(w - u)||_0.$$

By Corollary 1 there is a unit vector $\bar{u}$ in $R(E(\mu))$ such that

$$||w - \bar{u}||_0 \leqq C \, ||T - T_h||_{0,s_1}.$$

Hence

$$||[(\mu - T)^l - (\mu - T_h)^l]w||_0$$

(3.27) $$= \left|\left| \sum_{j=0}^{l-1} (\mu - T_h)^j(T_h - T)(\mu - T)^{l-j-1}[(w - \bar{u}) + \bar{u}] \right|\right|_0$$

$$\leqq C(||w - \bar{u}||_0 + ||T_h - T||_{0,s_1}) \leqq C \, ||T_h - T||_{0,s_1}.$$

Since $k \leqq l$,

$$||(\mu - T_h)^l w||_0 = \left|\left| \sum_{j=0}^{l} \binom{l}{j}(\mu - \mu(h))^j(\mu(h) - T_h)^{l-j}w \right|\right|_0$$

(3.28) $$= \left|\left| \sum_{j=l-k+1}^{l} \binom{l}{j}(\mu - \mu(h))^j(\mu(h) - T_h)^{l-j}w \right|\right|_0$$

$$\leqq C \, |\mu - \mu(h)|^{l-k+1}.$$

Using (3.26), (3.27), and (3.28), we have

$$||w - u||_0 \leq C \, ||[(\mu - T)^l - (\mu - T_h)^l]w + (\mu - T_h)^l w||_0$$

$$\leq C\{|\mu - \mu(h)|^{l-k+1} + ||T_h - T||_{0,s_1}\}.$$

The result now follows from Theorem 3.

4. Second-Order Boundary Value Problems. Let L be a second-order operator given by

$$L\varphi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial \varphi}{\partial x_j}\right) + \sum_{i=1}^{N} b_i \frac{\partial \varphi}{\partial x_i} + c\varphi$$

where a_{ij}, b_i and c are in $C^\infty(\bar{\Omega})$ and $a_{ij} = a_{ji}$. We will assume that L is uniformly strongly elliptic; i.e., there is a positive constant a_0 such that

$$\text{Re} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^{N} \xi_i^2$$

for all real $\xi_1, \cdots, \xi_N$ and all $x \in \Omega$. The sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$ associated with L is given by

$$B(\varphi, \psi) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \bar{\psi}}{\partial x_j} \, dx + \sum_{i=1}^{N} \int_\Omega b_i \frac{\partial \varphi}{\partial x_i} \bar{\psi} \, dx + \int_\Omega c\varphi\bar{\psi} \, dx.$$

Let $b = \max_{1 \leq i \leq N; x \in \Omega} |b_i(x)|$.

We shall assume without loss that $\text{Re } c \geq a_0/2 + b^2/2a_0$ since adding a constant to c only shifts the eigenvalues. Under this assumption B is coercive on $H^1(\Omega)$; i.e.,

(4.1) $$\text{Re } B(\varphi, \varphi) \geq \tfrac{1}{2}a_0 \, ||\varphi||_1^2$$

for all $\varphi \in H^1(\Omega)$. Clearly, because of the boundedness of the coefficients, we also have that B is continuous on $H^1(\Omega) \times H^1(\Omega)$; i.e., for a suitable constant a_1,

(4.2) $$|B(\varphi, \psi)| \leq a_1 \, ||\varphi||_1 \, ||\psi||_1$$

for all $\varphi, \psi \in H^1(\Omega)$.

Let $H_0^1(\Omega)$ be the closure in $H^1(\Omega)$ of the infinitely differentiable functions with compact support in Ω, $C_0^\infty(\Omega)$, and V any closed subspace of $H^1(\Omega)$ with $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. A class of boundary value problems associated with B may be formulated as follows: Given $f \in \mathcal{L}_2(\Omega)$ find $u \in V$ such that for all $\varphi \in V$,

(4.3) $$B(u, \varphi) = (f, \varphi).$$

With our assumption on B this problem always has a unique solution u (cf. [26]) and we denote the linear operator which takes f to u by T. In the case $V = H_0^1(\Omega)$ this is just the weak formulation of the Dirichlet problem and for $V = H^1(\Omega)$, T is the solution operator for the Neumann problem. We suppose now that V is either $H_0^1(\Omega)$ or $H^1(\Omega)$.

If, in addition, we have $f \in H^s(\Omega)$ for s real, the following estimate is valid:

(4.4) $$||Tf||_{s+2} \leq C_s \, ||f||_s$$

where C_s is an appropriate constant (cf. [35], [36]). The estimate (4.4) allows us to extend the operator T from $\mathcal{L}_2(\Omega)$ to $H^s(\Omega)$ for any $s < 0$. Now let $s_0 \geq 0$ be fixed

and consider the operator T as being defined on $H^{-s_0}(\Omega)$. Then (4.4) shows that

$$T: H^s(\Omega) \to H^{s+2}(\Omega)$$

as a bounded operator for $-s_0 \leq s$. Hence T as defined here satisfies the conditions set forth in Section 3.

An eigenvalue corresponding to the boundary value problem is a complex number λ such that

$$(4.5) \qquad\qquad B(u, \varphi) = \lambda(u, \varphi)$$

for some nonzero $u \in V$ and all $\varphi \in V$. An eigenvalue of T is a complex number μ such that $Tu = \mu u$ for some nonzero $u \in H^{-s_0}(\Omega)$. As was pointed out above, the spectrum of T consists of nonzero eigenvalues and zero. Clearly, for any eigenvalue μ of T we have that $\lambda = 1/\mu$ is an eigenvalue satisfying (4.5). Similarly, for λ an eigenvalue of (4.5), $\mu = 1/\lambda$ is an eigenvalue of T. It is also well known, because of the smoothness assumptions on the coefficients, that $Lu = \lambda u$ in Ω and, depending on the choice of V, u will satisfy certain boundary contitions. For $V = H_0^1(\Omega)$ we have

$$u = 0 \quad \text{on } \partial\Omega$$

and for $V = H^1(\Omega)$,

$$\partial u/\partial \nu = 0 \quad \text{on } \partial\Omega,$$

where $\partial/\partial\nu = \sum_{i,j=1}^N a_{i,j} n_j \partial/\partial x_i$ and n_j is the jth component of the outward unit normal on $\partial\Omega$.

Finally, we shall need in our applications some information about the $\mathfrak{L}_2(\Omega)$-adjoint T^* of $T|_{\mathfrak{L}_2(\Omega)}$. This operator is defined by

$$(Tf, \psi) = (f, T^*\psi)$$

for all $f, \psi \in \mathfrak{L}_2(\Omega)$ and satisfies $T^*\psi \in V$ and $B(\varphi, T^*\psi) = (\varphi, \psi)$ for all $\varphi \in V$.

In addition we have as for T, the regularity estimate for $s \geq 0$ and $\psi \in H^s(\Omega)$,

$$(4.6) \qquad\qquad ||T^*\psi||_{s+2} \leq C_s^* ||\psi||_s$$

with C_s^* an appropriate constant. We shall not need (4.6) with $s < 0$ even though it is valid.

5. Finite-Dimensional Subspaces of $H^k(\Omega)$.

Let $\{S_h\}_{0 < h \leq 1}$ be a one-parameter family of finite-dimensional vector spaces. For given integers k and r with $0 \leq k \leq r$ we shall say that $\{S_h\}_{0 < h \leq 1}$ is of class $S_{k,r}$ if $S_h \subset H^k(\Omega)$ for each h and if there is a constant C independent of h such that for any $v \in H^t(\Omega)$ with $k \leq t \leq r$,

$$(5.1) \qquad\qquad \inf_{\chi \in S_h} \sum_{j=0}^k h^j ||v - \chi||_j \leq Ch^t ||v||_t.$$

Analogously, we define the class $\overset{\circ}{S}_{k,r}$ to consist of those families $\{S_h\}_{0 < h \leq 1}$ such that for each h, $S_h \subset H^k(\Omega)$, and (5.1) is required to hold only for $v \in H^t(\Omega) \cap H_0^1(\Omega)$.

In the definition of $S_{k,r}$, condition (5.1) can be replaced by the condition that the same inequality be required to hold for $t = r$ only. This was proved in [12].

We note that there are many examples of families of class $S_{k,r}$. In particular the restriction to Ω of many different families of piecewise polynomial functions are of class $S_{k,r}$ for certain values of k and r (cf. [17], [23], [24], [39], [54], [55]). Each of the usual sets of trial functions used in conjunction with the finite element method (cf. [53]) is of class $S_{k,r}$ for some k and r. The simplest example of these for $N = 2$ is the family of restrictions to Ω of piecewise linear functions on a family of uniform triangulations of the plane with h the diameter of each triangle. Such a family is of class $S_{1,2}$ and hence satisfies (5.1) with $k = 1$ and $r = 2$.

Finally, by using tensor products of B-splines [37] it is easy to see that for any number of dimensions and any given values of k and r, families of class $S_{k,r}$ may be constructed. For such considerations, cf. [4], [5], [13], [14], [23].

6. Standard Galerkin Methods for the Dirichlet and Neumann Problems.

For $V = H^1(\Omega)$ let $\{S_h\}_{0<h\leq 1}$ be a family of spaces of class $S_{1,r}$. For $V = H_0^1(\Omega)$, let $\{S_h\}_{0<h\leq 1}$ belong to $S_{1,r}$ and, in addition, assume that for each h, $S_h \subset H_0^1(\Omega)$. We note that such families may be difficult to construct in practice for domains of general shape. Hence we will treat Dirichlet's problem again in the later sections by alternative techniques which do not require that $S_h \subset H_0^1(\Omega)$.

We want to consider the eigenvalues of the Galerkin problem; i.e., the complex numbers $\lambda(h)$ satisfying $B(w, \varphi) = \lambda(h)(w, \varphi)$ for some nonzero $w \in S_h$ and all $\varphi \in S_h$. These we want to consider as approximations to the eigenvalues of (4.5).

To this end we define now a family of operators $\{T_h\}_{0<h\leq 1}$ on $\mathcal{L}_2(\Omega)$ as follows: For each h set $T_h f = u_h$, where u_h is the unique solution in S_h of the equation

$$(6.1) \qquad\qquad B(u_h, \varphi) = (f, \varphi)$$

for all $\varphi \in S_h$. The inequality (4.1) guarantees the existence and uniqueness of u_h. Hence, for each h,

$$T_h: \mathcal{L}_2(\Omega) \rightarrow S_h \subset \mathcal{L}_2(\Omega)$$

and, since S_h is finite-dimensional, T_h is compact.

Let $\mu(h)$ be a nonzero eigenvalue of T_h. Then there is a nonzero element w of S_h such that $T_h w = \mu(h)w$. By the definition of T_h,

$$B(w, \varphi) = \frac{1}{\mu(h)} B(T_h w, \varphi) = \frac{1}{\mu(h)} (w, \varphi)$$

so that $\lambda(h) = 1/\mu(h)$ is an eigenvalue of the Galerkin problem. Conversely, since none of the eigenvalues $\lambda(h)$ can be zero, each $\mu(h) = 1/\lambda(h)$ is an eigenvalue of T_h. Hence we may compare the eigenvalues of (4.5) with the Galerkin approximations by comparing the eigenvalues of T with those of T_h.

In order to apply our theorems we shall obtain estimates for $T - T_h$. Let $f \in \mathcal{L}_2(\Omega)$ be arbitrary. Combining (4.3) and (6.1) we have that

$$(6.2) \qquad\qquad B((T - T_h)f, \varphi) = 0$$

for all $\varphi \in S_h$. Hence, using (4.1) and (6.2), it follows that

$$||(T - T_h)f||_1^2 \leq 2/a_0 \, |B((T - T_h)f, (T - T_h)f)|$$

$$= 2/a_0 \, |B((T - T_h)f, Tf - \chi)|$$

for any $\chi \in S_h$. Using the continuity of B, (4.2), it follows that for a suitable constant C,

$$||(T - T_h)f||_1 \leq C \inf_{\chi \in S_h} ||Tf - \chi||_1.$$

Now suppose that $f \in H^{t-2}(\Omega)$ for some t with $2 \leq t \leq r$. Then by (5.1) we have that

(6.3) $$||(T - T_h)f||_1 \leq Ch^{t-1} ||Tf||_t.$$

But for $f \in H^{t-2}(\Omega)$,

(6.4) $$||Tf||_t \leq C ||f||_{t-2},$$

where we have used the estimate (4.4) with $s = t - 2$. Combining (6.3) and (6.4), it follows that

(6.5) $$||(T - T_h)f||_1 \leq Ch^{t-1} ||f||_{t-2}$$

for $2 \leq t \leq r$.

Now we want to estimate norms of $(T - T_h)f$ with index lower than 1. To this end let $\psi \in C^\infty(\bar{\Omega})$ be arbitrary. We have that

(6.6) $$((T - T_h)f, \psi) = B((T - T_h)f, T^*\psi) = B((T - T_h)f, T^*\psi - \chi)$$

for any $\chi \in S_h$. Using (4.2) and (6.5) we obtain from (6.6),

(6.7) $$|((T - T_h)f, \psi)| \leq Ch^{t-1} ||f||_{t-2} ||T^*\psi - \chi||_1.$$

Now since χ is arbitrary we may take the infimum on the right-hand side of (6.7) and use (5.1) with $t = s + 2, 0 \leq s \leq r - 2$ to obtain

$$|((T - T_h)f, \psi)| \leq Ch^{s+t} ||f||_{t-2} ||T^*\psi||_{s+2}.$$

Using the estimate (4.6) we have that

$$|((T - T_h)f, \psi)| \leq Ch^{s+t} ||f||_{t-2} ||\psi||_s$$

for $0 \leq s \leq r - 2$ and $2 \leq t \leq r$. Since ψ is arbitrary in $C^\infty(\bar{\Omega})$ it follows that

$$||(T - T_h)f||_{-s} = \sup_{\psi \in C^\infty(\bar{\Omega})} |((T - T_h)f, \psi)|/||\psi||_s \leq Ch^{s+t} ||f||_{t-2}.$$

Now dividing by $||f||_{t-2}$ and taking the supremum over functions $f \in H^{t-2}(\Omega)$ we obtain

(6.8) $$||(T - T_h)||_{-s, t-2} \leq Ch^{s+t}$$

for $0 \leq s \leq r - 2$ and $2 \leq t \leq r$.

If we take $s = 0$ and $t = 2$ in (6.8) we obtain easily that (3.2) is satisfied. Hence Theorems 1–4 apply with $s_0 = s_1 = r - 2$. Thus let λ be any eigenvalue with algebraic multiplicity m satisfying (4.5) and $\mu = 1/\lambda$. Then the eigenvalues $\mu_1(h), \cdots, \mu_m(h)$ which converge to μ as $h \to 0$ are computed as the reciprocals of certain eigenvalues $\lambda_1(h), \cdots, \lambda_m(h)$ of the finite-dimensional eigenvalue problem

$$B(w, \varphi) = \lambda_i(h)(w, \varphi)$$

for nonzero $w \in S_h$ and all $\varphi \in S_h$. Theorem 2 thus yields the estimate

$$(6.9) \qquad \left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{m} 1/\lambda_j(h) \right)^{-1} \right| \leq Ch^{2r-2}.$$

We point out that in the case of a simple eigenvalue ($m = 1$) we have that $|\lambda - \lambda(h)| \leq Ch^{2r-2}$, and that this estimate yields the same rate of convergence obtainable in the case that the problem is selfadjoint using the methods of [11].

Using Theorem 1 with $s = 0$ we obtain for the invariant subspaces

$$(6.10) \qquad \max_{1 \leq i \leq m} ||u_i - w_i||_0 \leq Ch^r.$$

We note that in case the boundary or the coefficients do not possess the required smoothness properties we would obtain lower rates of convergence. In such a case this would indeed be expected.

7. Least Squares Method.

One method of approximating the solution of the Dirichlet problem using subspaces S_h which are not required to satisfy any boundary conditions is the least squares method of Bramble and Schatz [15], [16]. Although the case studied in [15] was that of real coefficients all relevant theorems from [15] are valid in the case of strongly elliptic operators with complex coefficients with the spaces taken to be complex.

Now let $\{S_h\}_{0 < h \leq 1}$ be of class $\mathbb{S}_{2,r}$ with $r \geq 4$. Then the least squares approximation to the solution of Dirichlet's problem is given as follows: For $f \in \mathcal{L}_2(\Omega)$ define u_h to be the unique element of S_h such that

$$||L_h - f||_0^2 + h^{-3}|u_h|^2 = \inf_{\chi \in S_h} \{||L\chi - f||_0^2 + h^{-3}|\chi|^2\}.$$

Here we have used the notation

$$|\varphi|^2 = \int_{\partial\Omega} |\varphi|^2 \, d\sigma,$$

with σ the measure on $\partial\Omega$ induced by the Lebesgue measure on $\mathbf{R}^{N-1}$. The corresponding inner product will henceforth be denoted by

$$\langle \varphi, \psi \rangle = \int_{\partial\Omega} \varphi\bar{\psi} \, d\sigma.$$

Because of uniqueness in the Dirichlet problem, (4.3) with $V = H_0^1(\Omega)$, $u_h \in S_h$ exists and is unique and we denote the corresponding linear solution operator by T_h. Hence we have that $T_h f = u_h$ and $T_h : \mathcal{L}_2(\Omega) \to S_h \subset \mathcal{L}_2(\Omega)$ and is compact.

Clearly, an equivalent characterization of u_h is given by

$$(Lu_h, L\varphi) + h^{-3}\langle u_h, \varphi \rangle = (f, L\varphi)$$

for all $\varphi \in S_h$. Now it is easy to see that $\mu(h)$ is a nonzero eigenvalue of T_h and $w \in S_h$ is a corresponding eigenfunction if and only if

$$(7.1) \qquad (Lw, L\varphi) + h^{-3}\langle w, \varphi \rangle = (w, L\varphi)/\mu(h)$$

for all $\varphi \in S_h$. Hence, again, we may compare the eigenvalues $\lambda(h) = 1/\mu(h)$ of the system (7.1) with those of (4.5) by comparing the eigenvalues of T with those of T_h.

Note that even in the case that L is selfadjoint, (7.1) gives rise to a nonsymmetric

matrix eigenvalue problem so that the convergence estimates for the eigenvalues do not follow from the standard techniques using the Rayleigh quotient. In particular, the technique of [11] does not apply.

In order to see that Theorems 1–4 apply, we need only give estimates for $T - T_h$ in various norms. But these follow directly from Corollary 4.1 of [15]. In Corollary 4.1, take $\gamma = \frac{3}{2}$, $\lambda = t - 2$, $g \equiv 0$ and $l = -s$. We have $e = (T - T_h)f$ and $\|(T - T_h)f\|_{-s} \leqq Ch^{s+t} \|f\|_{t-2}$ for $0 \leqq s \leqq r - 4$ and $2 \leqq t \leqq r$. This means that $\|(T - T_h)\|_{-s, t-2} \leqq Ch^{s+t}$ for $0 \leqq s \leqq r - 4$ and $2 \leqq t \leqq r$. Taking $s = 0$ and $t = 2$ we see immediately that (3.2) is satisfied. Now let λ be an eigenvalue of (4.5) with algebraic multiplicity m. Then the m eigenvalues $\lambda_1(h), \cdots, \lambda_m(h)$ which converge to λ satisfy

$$\left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{m} 1/\lambda_j(h) \right)^{-1} \right| \leqq Ch^{2r-4}.$$

This is just the result of Theorem 2 with $s_0 = r - 4$ and $s_1 = r - 2$.

The estimate (6.10) given by Theorem 1 is the same in this case.

8. Nitsche's Method for the Dirichlet Problem.

Although the least squares method requires nothing special about the elements of S_h at the boundary of Ω, it does involve second derivatives in the quadratic form and hence might lead to problems of conditioning of the resulting linear system.

Nitsche has proposed a method in [28] for approximating the solution of the Dirichlet problem for Poisson's equation. This method involves a quadratic form which is not bounded below on the whole space but which is positive definite on subspaces S_h which satisfy certain additional conditions. Because of this indefiniteness property the standard techniques for dealing with selfadjoint problems do not lead to a convergence proof or estimates for the rate of convergence of the corresponding eigenvalues.

Here we shall formulate Nitsche's method for the general nonselfadjoint equation with complex coefficients and prove the estimates necessary for the application of our theorems to the study of eigenvalues.

Let $\{S_h\}_{0 < h \leqq 1}$ be of class $\overset{\circ}{S}_{2,r}$ and, in addition, we assume that there is a constant C_1 such that

$$(8.1) \qquad \sum_{j=1}^{N} \int_{\partial \Omega} \left| \frac{\partial \varphi}{\partial x_j} \right|^2 d\sigma \leqq C_1 h^{-1} \|\varphi\|_1^2$$

for all $\varphi \in S_h$ and $0 < h \leqq 1$.

Define now the quadratic form

$$N_\gamma(\varphi, \psi) = B(\varphi, \psi) - \left\langle \varphi, \frac{\partial \psi}{\partial \bar{\nu}} - \sum_{j=1}^{N} b_j n_j \psi \right\rangle - \left\langle \frac{\partial \varphi}{\partial \nu}, \psi \right\rangle + \gamma h^{-1} \langle \varphi, \psi \rangle,$$

where γ is a positive constant, $\partial/\partial \nu = \sum_{i,j=1}^{N} a_{i,j} n_i \partial/\partial x_j$ (as in Section 4) and $\partial/\partial \bar{\nu} = \sum_{i,j=1}^{N} \bar{a}_{i,j} n_i \partial/\partial x_j$.

Let us define, for each h, the norm

$$|||\varphi||| = \left(\sum_{j=1}^{N} \left[\left\| \frac{\partial \varphi}{\partial x_j} \right\|_0^2 + h \left| \frac{\partial \varphi}{\partial x_j} \right|^2 \right] + h^{-1} |\varphi|^2 \right)^{1/2}$$

for $\varphi \in H^2(\Omega)$ with $\partial \varphi / \partial x_i$ on $\partial \Omega$ its trace. It is a simple consequence of (4.1) and Schwarz's inequality that there are constants C and γ_0 such that for $\gamma \geqq \gamma_0$,

$$(8.2) \qquad\qquad |||\varphi|||^2 \leqq C |N_\gamma(\varphi, \varphi)|$$

for all $\varphi \in S_h$. In this estimate explicit use is made of (8.1).

It is equally clear that N_γ is a sesquilinear form which is continuous with respect to $|||\cdot|||$; i.e., there is a constant C such that

$$(8.3) \qquad\qquad |N_\gamma(\varphi, \psi)| \leqq C |||\varphi||| \, |||\psi|||$$

for all φ and ψ in $H^2(\Omega)$. Now for a given $f \in \mathcal{L}_2(\Omega)$ let $u_h = T_h f$ be the unique solution of the equations

$$(8.4) \qquad\qquad N_\gamma(u_h, \varphi) = (f, \varphi)$$

for all $\varphi \in S_h$. The existence and uniqueness is assured by (8.2). Thus we have defined a family $\{T_h\}_{0 < h \leqq 1}$ of compact operators on $\mathcal{L}_2(\Omega)$. As in the previous sections the corresponding eigenvalues $\lambda(h)$ satisfy

$$(8.5) \qquad\qquad N_\gamma(w, \varphi) = \lambda(h)(w, \varphi)$$

for some nonzero w in S_h and all $\varphi \in S_h$. Clearly $\mu(h)$ is a nonzero eigenvalue of T_h if and only if $\lambda(h) = 1/\mu(h)$ is an eigenvalue of (8.5).

To apply our theorems to the comparison of the eigenvalues of (4.5) with those of (8.5) it again suffices to estimate certain norms of $T - T_h$ with T defined through (4.3) with $V = H_0^1(\Omega)$.

To this end we note first that

$$(8.6) \qquad\qquad N_\gamma(Tf, \varphi) = (f, \varphi)$$

for all $\varphi \in H^2(\Omega)$. Hence, combining (8.4) and (8.6), we obtain

$$(8.7) \qquad\qquad N_\gamma((T - T_h)f, \varphi) = 0$$

for all $\varphi \in S_h$. We first obtain an estimate for $|||(T - T_h)f|||$. By the triangle inequality we have that, for any $\chi \in S_h$,

$$(8.8) \qquad\qquad |||(T - T_h)f||| \leqq |||Tf - \chi||| + |||\chi - T_h f|||.$$

Since $\chi - T_h f \in S_h$, we may use (8.2) to obtain

$$(8.9) \qquad\qquad \begin{aligned} |||\chi - T_h f|||^2 &\leqq C |N_\gamma(\chi - T_h f, \chi - T_h f)| \\ &= C |N_\gamma(\chi - Tf, \chi - T_h f)|, \end{aligned}$$

the last equation coming from (8.7). Using (8.9) and (8.3) it follows that

$$(8.10) \qquad\qquad |||\chi - T_h f||| \leqq C |||\chi - Tf|||.$$

The last inequality, (8.10), together with (8.8) yields

$$(8.11) \qquad\qquad |||(T - T_h)f||| \leqq C \inf_{\chi \in S_h} |||\chi - Tf|||.$$

Now by standard trace inequalities it follows that for any $v \in H^2(\Omega)$,

$$(8.12) \qquad\qquad |||v||| \leqq C \sum_{j=0}^{2} h^{j-1} ||v||_j$$

for $0 < h \leqq 1$. Combining (8.11) and (8.12) and using the property (5.1) of S_h we obtain

$$|||(T - T_h)f||| \leqq Ch^{t-1} ||Tf||_t$$

for $2 \leqq t \leqq r$. It follows then from (4.4) that, if $f \in H^{t-2}(\Omega)$ then

$$(8.13) \qquad |||(T - T_h)f||| \leqq Ch^{t-1} ||f||_{t-2}.$$

To estimate the norms $||T - T_h||_{-s, t-2}$ for $0 \leqq s \leqq r - 2$ and $2 \leqq t \leqq r$ we proceed as in Section 6. Let $\psi \in C^\infty(\bar{\Omega})$ be arbitrary and note that $N_\gamma(\varphi, T^*\psi) = (\varphi, \psi)$ for all $\varphi \in H^2(\Omega)$. In particular,

$$((T - T_h)f, \psi) = N_\gamma((T - T_h)f, T^*\psi)$$

$$= N_\gamma((T - T_h)f, T^*\psi - \chi)$$

for all $\chi \in S_h$. From (8.3) it follows that

$$|((T - T_h)f, \psi)| \leqq C |||(T - T_h)f||| \, |||T^*\psi - \chi|||.$$

Using (8.13), (8.12) and the approximability assumptions (5.1) it follows that

$$|((T - T_h)f, \psi)| \leqq Ch^{s+t} ||f||_{t-2} ||T^*\psi||_{s+2}.$$

The estimate (4.7) yields

$$|((T - T_h)f, \psi)| \leqq Ch^{s+t} ||f||_{t-2} ||\psi||_s,$$

for $0 \leqq s \leqq r - 2$ and $2 \leqq t \leqq r$. This inequality implies immediately that

$$||T - T_h||_{-s, t-2} \leqq Ch^{s+t}$$

for $0 \leqq s \leqq r - 2$ and $2 \leqq t \leqq r$. In particular, with $s = 0$, $t = 2$, we have

$$\lim_{h \to 0} ||T - T_h||_{0,0} = 0$$

so that Theorems 1–4 apply. The results (6.9) and (6.10) are valid now in this case.

9. Nitsche's Method with Nearly Zero Boundary Conditions.

If the subspaces S_h satisfy, in addition to condition (8.1), a certain condition of smallness on the boundary then Nitsche [29] has observed that the form N_0 also leads to good approximation in the Dirichlet problem.

More precisely, suppose that there is a constant C_2 such that for all $\varphi \in S_h$,

$$(9.1) \qquad |\varphi|^2 \leqq C_2 h \, ||\varphi||_1^2.$$

Suppose further that $C_2 \leqq (a_0/8)(8N\tilde{a}^2 C_1/a_0 + \tilde{b})^{-1}$ where C_1 is the constant in (8.1) and $\tilde{a}$ and $\tilde{b}$ are constants such that

$$\max_{1 \leqq i \leqq N; x \in \partial\Omega} \left| \sum_{j=1}^N a_{ij} n_j \right| \leqq \tilde{a} \quad \text{and} \quad \max_{x \in \partial\Omega} \left| \sum_{j=1}^N b_j n_j \right| \leqq \tilde{b}.$$

This condition just says that (9.1) must hold with C_2 a sufficiently small constant. In fact, in many examples where a condition such as (9.1) is satisfied, it is also satisfied with $C_2 = C_2(h)$ and $C_2(h) = o(1)$ as $h \to 0$. In this connection see [29], [18], [55].

Now with this condition satisfied (8.2) is valid with $\gamma = 0$ and all subsequent considerations of Section 8 are valid in this case. Hence Theorems 1–4 again apply and the estimates (6.9) and (6.10) are valid in this case.

10. Lagrange Multiplier Method of Babuška.

Recently Babuška [6] has introduced a method for constructing an approximate solution of the Dirichlet problem in a subspace S_h which belongs to a family of class $\mathcal{S}_{1,r}$. He introduces analogous classes of spaces of approximating functions on $\partial\Omega$ and simultaneously obtains an approximation of $\partial u/\partial \nu$, the conormal derivative of the solution on $\partial\Omega$. As in the other examples this method gives rise to a family of operators which fits nicely into our theory for the approximation of eigenvalues. Hence, as we shall see, estimates similar to (6.9) and (6.10) are valid.

In order to describe precisely the method, we need some additional notation. For s any real number let $H^s(\partial\Omega)$ denote the Sobolev space of (complex valued) functions on $\partial\Omega$ (cf. [26]) and denote by $|\cdot|_s$ the corresponding norm.

In analogy with Section 5, let $\{S_h'\}_{0<h\leq 1}$ be a one-parameter family of vector spaces. For given real numbers k' and r' with $\frac{1}{2} \leq k' \leq r'$ we shall say that $\{S_h'\}_{0<h\leq 1}$ is of class $\mathcal{S}_{k',r'}'$ if $S_h' \subset H^{k'}(\partial\Omega)$ for each h and if there is a constant C independent of h such that for any $v \in H^s(\partial\Omega)$ and $-\frac{1}{2} \leq l \leq k' \leq s \leq r'$,

(10.1) $$\inf_{\chi \in S_h'} |v - \chi|_l \leq Ch^{s-l} |v|_s$$

and

(10.2) $$|\varphi|_{1/2} \leq Ch^{-1} |\varphi|_{-1/2}$$

holds for all $\varphi \in S_h'$.

An example of a situation in which such a family may be easily constructed is as follows: Let $\Omega \subset \mathbf{R}^2$ and $\partial\Omega$ be a simple closed (smooth) curve. By parametrizing $\partial\Omega$ we may easily construct a family of periodic complex valued spline functions corresponding to a family of uniform subdivisions of $\partial\Omega$ of length h. If we take $S_h' \subset H^1(\partial\Omega)$ for each h then the condition (10.2) follows easily from the property $|\varphi|_1 \leq Ch^{-1} |\varphi|_0$ which is easy to verify in view of the uniformity of the subdivisions. In addition, it is not difficult to see that the validity of (10.1) with $0 \leq l \leq k'$ implies (10.1) for $-\frac{1}{2} \leq l \leq k'$.

Now let $H = H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ and let $U = \{u, u'\} \in H$. (Here $u \in H^1(\Omega)$ and $u' \in H^{-1/2}(\partial\Omega)$ are, of course, independent and $\|U\|_H = (\|u\|_1^2 + |u'|_{-1/2}^2)^{1/2}$.) On $H \times H$ consider the sesquilinear form

$$\mathcal{B}(U, W) = B(u, w) - \langle u, w' \rangle - \langle u', w \rangle,$$

where B is the form introduced in Section 4. Now let $\{\mathfrak{M}_h\}_{0<h\leq 1}$ be a family of subspaces of H with $\mathfrak{M}_h = S_{\delta h} \times S_h'$, δ a fixed small constant, $\{S_h\}_{0<h\leq 1}$ of class $\mathcal{S}_{1,r}$ and $\{S_h'\}_{0<h\leq 1}$ of class $\mathcal{S}_{1/2,r-3/2}'$ with $r \geq 2$. Babuška [6] has proved the following inequality:

(10.3) $$\sup_{\Phi \in \mathfrak{M}_h; \, \|\Phi\|_H \leq 1} |\mathcal{B}(U, \Phi)| \geq C \, \|U\|_H$$

for all $U \in \mathfrak{M}_h$, for a suitable positive constant C independent of h and U provided that δ is sufficiently small. It is by means of this inequality that we may study a family of approximate solutions to (4.3) with $V = H_0^1(\Omega)$ (the Dirichlet problem).

We first note that if $f \in \mathcal{L}_2(\Omega)$ and u is the solution of (4.3), then for $U = \{u, \partial u/\partial \nu\}$ and $\Phi = \{\varphi, \varphi'\}$ arbitrary

$$(10.4) \qquad \mathcal{B}(U, \Phi) = B(u, \varphi) - \langle \partial u/\partial \nu, \varphi \rangle = (f, \varphi).$$

By virtue of (10.3) there exists a unique $U_h \in \mathfrak{M}_h$ satisfying

$$(10.5) \qquad B(U_h, \Phi) = (f, \varphi)$$

for all $\Phi \in \mathfrak{M}_h$. Now for $U_h = \{u_h, u_h'\}$ define $T_h f = u_h$. Then, $T_h : \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$ and, since $S_{\delta h}$ is finite dimensional, T_h is compact. As in Section 6 let $\mu(h)$ be a nonzero eigenvalue of T_h; i.e.,

$$(10.6) \qquad T_h w = \mu(h)w$$

for some nonzero $w \in S_h$. Then with $\lambda(h) = 1/\mu(h)$ and $W = \{w, w'\}$ for a certain $w' \in S_h'$, we have

$$(10.7) \qquad \mathcal{B}(W, \Phi) = \lambda(h)(w, \varphi)$$

for all $\Phi \in \mathfrak{M}_h$. Hence, if we choose bases for $S_{\delta h}$ and S_h', then (10.7) reduces to a matrix eigenvalue problem with $\lambda(h)$ an eigenvalue. Conversely for $\lambda(h)$ satisfying (10.7) with w a nonzero element of $\mathfrak{M}_h$ it follows easily that (10.6) holds with $\mu(h) = 1/\lambda(h)$. Hence the nonzero eigenvalues $\mu(h)$ of (10.6) may be determined from a finite-dimensional (matrix) eigenvalue problem.

Now to compare the eigenvalues of T with those of T_h we shall estimate the appropriate norms of $T - T_h$. Babuška [6] has already obtained some of the necessary estimates. Set $E = \{e, e'\}$ where $e = u - u_h$ and $e' = \partial u/\partial \nu - u_h'$. He has shown that

$$(10.8) \qquad ||E||_H \leqq Ch^{t-1} ||f||_{t-2}$$

and

$$(10.9) \qquad ||e||_0 \leqq Ch^t ||f||_{t-2},$$

for $2 \leqq t \leqq r$.

The method used for obtaining (10.9) also leads to the remaining desired estimates which, for completeness, we derive here.

Let $\psi \in C^\infty(\bar{\Omega})$ and set $V = \{T^*\psi, \partial T^*\psi/\partial \bar{\nu}\}$. We have that $(e, \psi) = \mathcal{B}(E, V)$ and, using (10.4) and (10.5), $(e, \psi) = \mathcal{B}(E, V - \Phi)$ for all $\Phi \in \mathfrak{M}_h$. Now by the trace inequality (cf. [26]), $|v|_{1/2} \leqq C ||v||_1$ for $v \in H^1(\Omega)$, it follows immediately that $\mathcal{B}$ is continuous on $H \times H$ and hence, since Φ is arbitrary,

$$|(e, \psi)| \leqq C ||E||_H \inf_{\Phi \in \mathfrak{M}_h} ||V - \Phi||_H.$$

It is an immediate consequence of the properties of $\mathfrak{M}_h$, (5.1) and (10.1), that for $0 \leqq s \leqq r - 2$,

$$|(e, \psi)| \leqq Ch^{s+1}(||T^*\psi||_{s+2} + |\partial T^*\psi/\partial \bar{\nu}|_{s+1/2}) ||E||_H.$$

The trace inequality

$$|\partial T^*/\partial \bar{\nu}|_{s+1/2} \leqq C ||T^*\psi||_{s+2}$$

with $s \geqq 0$ is also well known (cf. [26]) so that

$$|(e, \psi)| \leqq Ch^{s+1} \, ||T^*\psi||_{s+2} \, ||E||_H.$$

We have then from (4.6) and (10.8) that

$$|(e, \psi)| \leqq Ch^{t+s} \, ||\psi||_s \, ||f||_{t-2}.$$

Clearly, since $e = (T - T_h)f$, we have that

$$||T - T_h||_{-s,\,t-2} \leqq Ch^{t+s}$$

for $0 \leqq s \leqq r - 2$ and $2 \leqq t \leqq r$. Hence, in particular, setting $s = 0$ and $t = 2$, we see that

$$\lim_{h \to 0} ||T - T_h||_{0,0} = 0$$

so that Theorems 1–4 apply. Again the estimates (6.9) and (6.10) are valid.

11. $\mathcal{L}_2$-Projections. In each of the foregoing examples the approximate operators T_h were chosen to be of the form P_hT where P_h is some projection onto S_h. The crucial point in all these Rayleigh-Ritz-Galerkin methods is that although the operator T is usually not explicitly known, the operator P_h is chosen in such a way that P_hT can be explicitly evaluated. In case T is an operator which can be evaluated, for example, if T were given as an integral operator whose kernel is a known Green's function, then a possible choice of the operator P_h is just the orthogonal projection in $\mathcal{L}_2(\Omega)$ onto S_h.

As a final example we suppose that T is an operator of the general type described in Section 3 and we take $T_h = P_hT|_{\mathcal{L}_2(\Omega)}$, where P_h is the orthogonal projection in $\mathcal{L}_2(\Omega)$ onto S_h with $\{S_h\}_{0 < h \leqq 1}$ of class $S_{0,r}$. Assume that $r \geqq \epsilon$ where ϵ is as in Section 3.

To obtain estimates for $T - T_h$ let $f \in H^{r-\epsilon}(\Omega)$ and $\psi \in C^\infty(\bar{\Omega})$. Then

$$|((T - T_h)f, \psi)| = |((I - P_h)Tf, (I - P_h)\psi)|$$

$$\leqq \inf_{\chi_1 \in S_h} ||Tf - \chi_1||_0 \inf_{\chi_2 \in S_h} ||\psi - \chi_2||_0.$$

It follows from (5.1) and (3.1) that

$$|((T - T_h)f, \psi)| \leqq Ch^{s+t} \, ||\psi||_s \, ||f||_{t-\epsilon}$$

for $0 \leqq s, t \leqq r$. Hence

$$||T - T_h||_{-s,\,t-\epsilon} \leqq Ch^{s+t}$$

for $0 \leqq s, t \leqq r$ and in particular choosing $s = 0$ and $t = \epsilon$ it follows that

$$\lim_{h \to 0} ||T - T_h||_{0,0} = 0.$$

Thus Theorems 1–4 apply. Let μ be an eigenvalue of T with algebraic multiplicity m and $\hat{\mu}(h) = (1/m) \sum_{i=1}^m \mu_i(h)$, where the $\mu_i(h)$'s are the m eigenvalues of T_h which converge to μ as $h \to 0$. Then from Theorem 2 we find that, for $0 < h \leqq h_2$,

$$|\mu - \hat{\mu}(h)| \leqq Ch^{2r}.$$

As before, for $\{u_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^m$ the respective orthonormal bases for the corresponding invariant subspaces, we have from Theorem 1 that

$$\max_{1 \le i \le m} ||u_i - w_i||_0 \le Ch^r.$$

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