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Convergence for a Vortex Method for Solving Euler's Equation*

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Abstract. We consider a new vortex approximation for solving the initial-value problem for the Euler equations in two dimensions. We assume there exists a smooth solution to these equations and that the vorticity has compact support. Then we show that our approximation to the velocity field converges uniformly in space and time for a short time interval.

1. Introduction. The flow of an incompressible inviscid fluid is described by the Euler equations. In two dimensions the initial-value boundary free problem for the Euler equations may be written in the form

$$\xi_t + (\mathbf{u}\cdot\nabla)\xi = 0, \tag{1.1a}$$

$$\Delta\psi = -\xi, \tag{1.1b}$$

$$u = -\partial_y\psi, \qquad v = \partial_x\psi, \tag{1.1c}$$

where $\mathbf{u} = (u, v)$ is the velocity, ξ is the vorticity, and ψ the stream function, subject to initial data $\xi_0(x, y) = \xi(x, y; 0)$. Using (1.1), we may write the velocity in terms of the vorticity (see Batchellor [1, p. 527]).

$$u(x, y, t) = K(x, y) * \xi(x, y, t), \tag{1.2a}$$

$$v(x, y, t) = L(x, y) * \xi(x, y, t), \tag{1.2b}$$

where

$$K(x, y) = -\frac{1}{2\pi}\frac{y}{r^2}, \qquad L(x, y) = \frac{1}{2\pi}\frac{x}{r^2},$$

where $r^2 = x^2 + y^2$, and $*$ denotes convolution in the (x, y) plane.

One method of solution to these equations, originally suggested by Rosenhead [12], is the point-vortex method. Takami [13] found the approximation of a vortex sheet by point vortices and their subsequent evolution by Rosenhead's method did not appear valid. Moore [11] has carried this approach further and claims the method unreliable, no matter how many vortices are used and how accurately the integrations

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are performed. On the other hand, Christiansen [7] reports some success with his calculations.

Chorin [5], [6] has devised a modification of the point-vortex approximation for the full Navier-Stokes equations with boundaries and small viscosity and has made several accurate calculations with it. The purpose of our paper is to demonstrate rigorously the correctness of this approximation in the specialized case of inviscid smooth flow with no boundary.

2. Method of Solution and Notations. Let us review the standard point-vortex method in its simplest form. Assume the vorticity ξ consists of point masses of constant strength, i.e.,

$$\xi(z) = \sum_{i=1}^{N} k_i \delta(z - z_i(t)) \tag{2.1}$$

where δ is the Dirac delta function, and z, z_i are points in the plane. Substituting ξ of the form (2.1), we obtain (see Batchellor [1, pp. 527–532]) the following equation for $\mathbf{u} = (u, v)$, at each of the N points $z_j = (x_j, y_j)$,

$$\frac{dx_j}{dt} = -\frac{1}{2\pi} \sum_{i \neq j} \frac{k_i(y_j - y_i)}{|z_i - z_j|^2}; \qquad \frac{dy_j}{dt} = \frac{1}{2\pi} \sum_{i \neq j} \frac{k_i(x_j - x_i)}{|z_i - z_j|^2} \tag{2.2}$$

for $j = 1, \cdots, N$. To compute a solution of the Euler equation for the boundary free case, one solves the equations (2.2) subject to some initial distribution of vorticity of the type (2.1) which approximates an actual distribution of vorticity. To see how this method breaks down, consider an approximation to a vortex line consisting of a large number of point vortices. Following Eqs. (2.2), it is not hard to see that the vortices will wind around each other because of the singularities in the right-hand side (for a reproduction of Takami's results, see Moore [11]). The standard point-vortex method fails because of a spurious interaction of vortices at close range.

Chorin [4] first suggested, according to a theory of turbulence, that "blobs," rather than points of vorticity be considered. The stream function for a point vortex is $(\log |r|)/2\pi$ where r is the distance to the vortex. The stream function for a circular blob of vorticity $\xi = 1/2\pi r$ for $r < \delta$ and 0 vorticity elsewhere is

$$\begin{aligned} \psi_\delta(x, y) &= \frac{1}{2\pi} \log |r|, \qquad |r| \geqq \delta, \\ &= \frac{1}{2\pi} \frac{r}{\delta}, \qquad |r| < \delta. \end{aligned} \tag{2.3a}$$

Chorin's approach, then, amounts to a "cut-off" of the stream function of a point vortex. He has also used other cut-off's, including

$$\begin{aligned} \psi_\delta(x, y) &= \frac{1}{2\pi} \log |r|, \qquad |r| \geqq \delta, \\ &= \frac{1}{4\pi} \frac{r^2}{\delta^2}, \qquad |r| < \delta, \end{aligned} \tag{2.3b}$$

which corresponds to a blob of constant vorticity $\xi = 1/\pi\delta^2$ of radius δ. The importance of the exact form of the cut-off comes in using blobs to approximate vorticity generated at the boundary for the full Navier-Stokes equations. For the boundary free case, our proofs will depend on the fact that ψ_δ and its first derivatives inside $|r| < \delta$ are no larger than those at the boundary $|r| = \delta$. Cut-off's have been used experimentally and have been observed to improve greatly the accuracy of calculations made with a point-vortex approximation (see Birdsall et al. [2]). The formulation of the cut-off stream function in terms of the stream function of a small blob of vorticity suggests the analysis below. Without such a formulation, we appear to be introducing an *ad hoc* device, and we suggest that it is for this reason that such devices were previously regarded as experimental accidents, rather than predictable results.

Using (2.3a), we define the approximating kernels

$$\begin{aligned} K_\delta &= -\partial_y \psi_\delta = K(x, y), && r \geqq \delta, \\ &= -y/2\pi r\delta, && r < \delta, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} L_\delta &= \partial_x \psi_\delta = L(x, y), && r \geqq \delta, \\ &= x/2\pi r\delta, && r < \delta. \end{aligned} \tag{2.4b}$$

We divide the vorticity into small nonoverlapping blobs B_j and choose points $z_j = (x_j, y_j) \in B_j$ at $t = 0$. Let $z_j(t)$ denote the position of z_j at time t under the flow. Since by Kelvin's circulation theorem

$$\iint_{B_j} \xi = k_j \tag{2.5}$$

is a constant independent of time, we may then approximate the velocity (u, v) at the point z as follows:

$$\begin{aligned} u(z) = K * \xi(z) &= \iint K(z - z')\xi(z')\, dz' \approx \iint K_\delta(z - z')\xi(z')\, dz'' \\ &= \sum \iint_{B_j} K_\delta(z - z')\xi(z')\, dz' \\ &\approx \sum K_\delta(z - z_j)k_j. \end{aligned} \tag{2.6a}$$

We similarly approximate

$$v(z) \approx \sum L_\delta(z - z_j)k_j. \tag{2.6b}$$

Let the time step be Δt. Using the last line of (2.6a) to approximate $u = dx/dt$ and (2.6b) to approximate $v = dy/dt$, we move the points $z_i(0)$ to positions $\tilde{z}_i(n\Delta t)$ according to the scheme ($\tilde{z}_i = (\tilde{x}_i, \tilde{y}_i)$)

$$\begin{aligned} x_i((n+1)\Delta t) - x_i(n\Delta t) &= \Delta t\Big(\sum K_\delta(\tilde{z}_i(n\Delta t) - \tilde{z}_j(n\Delta t))k_j\Big), \\ \tilde{y}_i((n+1)\Delta t) - \tilde{y}_i(n\Delta t) &= \Delta t\Big(\sum L_\delta(\tilde{z}_i(n\Delta t) - \tilde{z}_j(n\Delta t))k_j\Big), \\ \tilde{x}_i(0\Delta t) = x_i(0), &\qquad \tilde{y}_i(0\Delta t) = y_i(0), \end{aligned} \tag{2.7}$$

where K_δ and L_δ are given by (2.4) and k_j are given by (2.5), and the summations are over all blobs B_j. This is our scheme which is to approximate the flow in (1.1).

Notations. We use the abbreviation $z = (x, y)$ for $(x, y) \in R^2$. For $\psi(z, t)$ continuous, let supp $\psi(z, t)$ be the support of ψ as a function of z for each t, i.e., supp $\psi(z, t)$

$=$ closure $\{z \mid \psi(z, t) \neq 0\}$. For a bounded set $B \subset R^2$, let $\operatorname{diam}(B) = \sup d(z_1, z_2)$, $z_1, z_2 \in B$, $d =$ Euclidean distance, and let $d(z, B) = \inf d(z, z')$, $z' \in B$. Let $|B|$ denote the area of B. For $\psi = (x, y, t)$, the space C^s will be the space of s times differentiable functions; the space C_0^s will be those ψ in C^s with compact support with seminorms $||\psi||_{\beta_1,\beta_2} = \max \sup |D^\alpha u|$, $\alpha_1 + \alpha_2 = \beta_1$, $\alpha_3 = \beta_3$ where if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ we denote

$$D^\alpha u = \partial^{|\alpha|} u / \partial x^{\alpha_1} \partial y^{\alpha_2} \partial t^{\alpha_3}.$$

For $\beta_1 = 0$, $\beta_2 = 0$, we use the conventional notation $||\psi||_\infty = ||\psi||_{0,0} = \sup |\psi|$.

We shall use the abuse of notation $\sum_{B_i \in A} (\cdots i \cdots)$ to mean summation over the index i such that $B_i \in A$. For r a real number $[r]$ will denote the integral part of r.

3. **Preliminaries: The Kernels K_δ and L_δ.** Throughout the remainder of this paper we will be discussing the approximation (2.7) to the Euler equations (1.1) in a fixed time interval $[0, \alpha_0]$. We assume furthermore that a smooth solution to (1.1) exists so that the velocity field (u, v) is in C^2 and the vorticity lies in C_0^1.

Our approximation depends upon dividing up the fluid into small blobs and accurately following them. In order to make precise those ideas we shall need some preliminary results. We have first a lemma about the extent of contraction and expansion in a smooth flow:

LEMMA 3.1. *Let $z_1(t)$, $z_2(t)$ be two points of the flow. Let the velocity field* $\mathbf{u} \in C^2$. *Then there exists a constant C so that*

$$C^{-1}|z_1(0) - z_2(0)| \leqq |z_1(t) - z_2(t)| \leqq C|z_1(0) - z_2(0)| \tag{3.1}$$

for $0 \leqq t \leqq \alpha_0$. The constant C depends on $||\mathbf{u}||_{2,0}$ for $0 \leqq t \leqq \alpha_0$.

Proof. Let $z_i(t) = (x_i(t), y_i(t))$, $u(z_i(t)) = u_i(t)$, $v(z_i(t)) = v_i(t)$ for $i = 1, 2$. Integrating the equations of motion, we have

$$x_1(t) - x_2(t) = x_1(0) - x_2(0) + \int_0^t (u_1(s) - u_2(s))\, ds, \tag{3.2a}$$

$$y_1(t) - y_2(t) = y_1(0) - y_2(0) + \int_0^t (v_1(s) - v_2(s))\, ds. \tag{3.2b}$$

For $0 \leqq s \leqq t$, we have

$$u_1(s) - u_2(s) = \frac{\partial u}{\partial x}(\eta(s))(x_1(s) - x_2(s)) + \frac{\partial u}{\partial y}(\eta(s))(y_1(s) - y_2(s))$$

where $\eta(s)$ is chosen by the mean value theorem. Express $v_1(s) - v_2(s)$ similarly and let $f(t) = x_1(t) - x_2(t)$, $g(t) = y_1(t) - y_2(t)$. Then by differentiating (3.2), we obtain

$$f'(t) = a(t)f(t) + b(t)g(t), \qquad g'(t) = c(t)f(t) + d(t)g(t),$$

where a, b, c, d are bounded in absolute value by a constant $m = \sup_{0 \leqq t \leqq T} ||u||_{2,0}$.

Then consider $|z_1(t) - z_2(t)|^2 = f^2(t) + g^2(t) = F(t)$. We find, since $\frac{1}{2}F'(t) = ff' + gg' = af^2 + (b + c)fg + dg^2$, that $-4mF \leqq F' \leqq 4mF$.

Integrating, we obtain (3.1) with $C = e^{2mT}$.

Let the region of vorticity be divided into a finite number of nonoverlapping regions B_i at $t = 0$. The B_i then move with the fluid, and from Lemma 3.1, we have

LEMMA 3.2. *Let $u \in C^2$ and $\xi \in C_0^1$. Then we may choose the B_i so that*

$$\max \operatorname{diam}(B_i) \leqq C\Delta t \tag{3.3}$$

for $0 \leqq t \leqq \alpha_0$ for some constant C independent of Δt and depending on α_0.

Proof. We choose $\max \operatorname{diam}(B_i) \leqq \Delta t$ at $t = 0$ and apply Lemma 3.1.

Remark. We use the hypothesis of the smoothness of the flow strongly to get the regions B_i to maintain their initial size, at least in order of magnitude. For our problem, the smoothness is a reasonable hypothesis on physical grounds, either in the problem of inviscid fluid flow or the problem of plasma flow (see Levy and Hockney [10]).

We shall need some technical results about sums of the form $\sum K_\delta(z - z_i)k_i$ where the z_i are near to B_i. The first of these is

LEMMA 3.3. *Let $d(z_i, B_i)$, the distance from z_i to B_i, be $O(\Delta t)$ for each i. Then we have*

$$\sum_i |K_\delta(z - z_i)|\,|B_i| = O(|\operatorname{supp}(\xi)|) = O(1) \tag{3.4}$$

independent of Δt and the same formula with K_δ replaced by L_δ.

Remark. Note that Eq. (3.4) is the analogue of the integral $\iint |K_\delta|$, where the integral is taken over $\operatorname{supp}(\xi)$.

Proof of Lemma 3.3. We may assume without loss of generality that $z = 0$ and that $d(z, \operatorname{supp} \xi) \leqq 1 + \max d(z_i, B_i)$. If the above does not hold, then $|K_\delta(z - z_i)| \leqq 1$ and (3.4) is bounded by $\sum |B_i| = |\operatorname{supp} \xi| = O(1)$. Let C_1 be a constant satisfying (3.3) of Lemma 3.2. Then let C_2 be a constant so that $d(z_i, B_i) \leqq C_2\Delta t$ for each i. We divide the z-plane into annuli using circles with a common center at the origin and radii $r_n = n\Delta r$, where $\Delta r = C_3\Delta t$, $C_3 = \max\{C_1, C_2\}$. Let $A_n = \{z \mid r_{n-1} \leqq |z| < r_n\}$ and divide the blobs B_j into classes I_n so that $B_j \in I_n$ if $r_{n-1} \leqq d(z, B_j) < r_n$, i.e., if the nearest point of B_j is in A_n.

We then have, for $B_j \in I_n$, $z_j \in A_{n-1} \cup A_n \cup A_{n+1}$ and therefore $|K_\delta(z - z_j)| \leqq (2\pi r_{n-2})^{-1}$ for $n \geqq 3$. Since the B_j are nonoverlapping, and since a blob $B_j \in I_n$ can extend, at most from A_n to A_{n+1},

$$\sum_{B_j \in I_n} |B_j| \leqq |A_n| + |A_{n+1}| = \pi(r_{n+1}^2 - r_{n-1}^2).$$

For $B_j \in I_1 \cup I_2$, $|K_\delta(z - z_j)| \leqq (2\pi\Delta r)^{-1}$ and $\sum_{B_j \in I_1 \cup I_2} |B_j| \leqq \pi(3\Delta r)^2$. Using these facts, we may write, for $N = [(\operatorname{diam}(\operatorname{supp} \xi) + 1)/\Delta r] + 1$,

$$\begin{aligned} \sum |K_\delta(z - z_i)|\,|B_i| &= \sum_{n=1}^{2} \sum_{B_j \in I_n} |K_\delta(z - z_j)|\,|B_j| + \sum_{n=3}^{N} |K_\delta(z - z_j)|\,|B_j| \\ &\leqq (2\pi\Delta r)^{-1} 9\pi(\Delta r)^2 + \sum_{n=3}^{N} (2\pi r_{n-2})^{-1}(r_{n-1} + r_{n+1})2\Delta r \\ &\leqq C\left(\Delta r + \sum_{n=3}^{N} (r_{n-2})^{-1}(r_{n+1} + r_{n-1})\Delta r\right) \\ &\leqq C(\operatorname{diam}(\operatorname{supp} \xi) + 1) = O(1). \end{aligned} \tag{3.5}$$

We also have

LEMMA 3.4. *Let $z_i \in B_i$ and let $|z_i - z_i'| \leqq C\Delta t$ for each i. Then, we have*

$$\sum |K_\delta(z - z_i) - K(z - z_i')|\,|B_i| = O(\log \Delta t) \max_i |z_i - z_i'| \tag{3.6}$$

and the same formula with K_δ replaced by L_δ.

Proof. Our proof is like that of Lemma 3.3. We use the fact that $|\partial K_\delta/\partial x|$ and $|\partial K_\delta/\partial y|$ are majorized by $(2\pi r^2)^{-1}$, and that the integral of this function in the annuli around the origin bounded by $r = \Delta t$ and $r = 1/\Delta t$ is $4\pi\,|\log \Delta t|$.

We divide the blobs as we did in the proof of Lemma 3.3, i.e., so that $B_i \in A_n$ if the nearest point of B_i is in A_n and assume $z = 0$. Let us recall that if $B_i \in A_n$, then $z_i' \in A_{n-1} \cup A_n \cup A_{n+1}$. Hence, we may majorize the left-hand side of (3.6) as follows:

$$\begin{aligned}&\sum_{n=1}^{2} \sum_{B_i \in I_n} (|K_\delta(z - z_i)| + |K_\delta(z - z_i')|)\,|B_i| \\ &\qquad + \sum_{n=3}^{N} \sum_{B_i \in I_n} \left(\frac{\partial K_\delta}{\partial x}(a_i)(x_i - x_i') + \frac{\partial K_\delta}{\partial y}(a_i)(y_i - y_i')\right) |B_i|\end{aligned} \tag{3.7}$$

where $z_i = (x_i, y_i)$ and $z_i' = (x_i', y_i')$ and a_i is chosen by the mean value theorem between z_i and z_i'. The first sum in (3.7) is $O(\Delta t)$ by the proof of Lemma 3.3 (see Eq. (3.5)). To estimate the second sum, we take absolute values and apply the triangle inequality to majorize (3.7) by

$$\sum_{n=3}^{N} \sum_{B_i \in I_n} \left(\left|\frac{\partial K_\delta}{\partial x}(a_i)\right| + \left|\frac{\partial K_\delta}{\partial y}(a_i)\right|\right) |B_i| \max_i |z_i - z_i'| + O(\Delta t).$$

For $B_i \in I_n$, we have

$$\max\left\{\left|\frac{\partial K_\delta}{\partial x}(a_i)\right|, \left|\frac{\partial K_\delta}{\partial y}(a_i)\right|\right\} \leqq (2\pi r_{n-2}^2)^{-1} \quad \text{and} \quad \sum_{B_i \in I_n} |B_i| \leqq \pi(r_{n+1}^2 - r_{n-1}^2).$$

Let us now assume $d(z, \operatorname{supp} \xi) \leqq (\Delta t)^{-1}$. We may then majorize the above by

$$\left(\sum_{n=3}^{N'} \frac{\pi(2r_{n+1})}{2\pi r_{n-2}^2} \Delta r\right) \max_i |z_i - z_i'| + O(\Delta t), \tag{3.8}$$

where $N' = \max\{N, (\Delta t)^{-1}\}$. But since $\int_A\int r^{-2}(r\,dr\,d\theta) = 4\pi\,|\log \Delta t|$ where A is the region bounded by $r = \Delta t$ and $r = (\Delta t)^{-1}$, we may bound the sum in (3.8) above by

$$O(\log \Delta t) \max |z_i - z_i'| + O(\Delta t). \tag{3.9}$$

Then since we will be able to assume $\max |z_i - z_i'|$ will be of order at least Δt, (3.9) is of the order $|\log \Delta t| \max_i |z_i - z_i'|$.

If $d(z, \operatorname{supp} \xi) > (\Delta t)^{-1}$, then we may majorize (3.6) by

$$\begin{aligned}&\sum_i (|K_\delta(z - z_i)| + |K_\delta(z - z_i')|)\,|B_i| \\ &\qquad \leqq \Delta t \sum |B_i| = O(\Delta t) = O(\log \Delta t) \max_i |z_i - z_i'|.\end{aligned}$$

Using Lemma 3.4, we may prove the following theorem about approximating the velocity field.

THEOREM 3.1. *Let $\mathbf{u} \in C^2$ and $\xi \in C_0^1$. Let B_i be chosen according to Lemma 3.2, and let $\delta = O(\Delta t)$.*

(3.10a) $$u(z) = \sum K_\delta(z - z_i)k_i + O(\log \Delta t(\max d(z_i, B_i))),$$

(3.10b) $$v(z) = \sum L_\delta(z - z_i)k_i + O(\log \Delta t(\max d(z_i, B_i))).$$

Proof. We will prove (3.10a). We observe that

(3.11)
$$\begin{aligned} u(z) &= \iint K(z - z')\xi(z')\,dz' \\ &= \iint K(z - z')\xi(z')\,dz' + O(\delta) \\ &= \sum \int_{B_i}\int K_\delta(z - z')\xi(z')\,dz' + O(\delta) \\ &= \sum K_\delta(z - z_i')\xi(z_i')\,|B_i| + O(\delta). \end{aligned}$$

The points $z_i' \in B_i$ in (3.11) are chosen by the mean value theorem for the integral. Noting that we then have, writing $k_i = \xi(z_i'')\,|B_i|$, for some z_i'' chosen by the mean value theorem,

(3.12)
$$\begin{aligned} u(z) = &\sum K_\delta(z - z_i)k_i \\ &+ \sum (K_\delta(z - z_i')\xi(z_i') - \sum K_\delta(z - z_i)\xi(z_i''))\,|B_i| + O(\Delta t) \\ = &\sum K_\delta(z - z_i)k_i + \sum (K_\delta(z - z_i') - K_\delta(z - z_i))\xi(z_i'')\,|B_i| \\ &+ \sum K_\delta(z - z_i')(\xi(z_i') - \xi(z_i''))\,|B_i| + O(\Delta t). \end{aligned}$$

We estimate the second term of (3.12) using the fact that ξ is continuous, of compact support, hence bounded. Therefore this term is of order $|\log \Delta t| \max_i |z_i - z_i'|$ using Lemma 3.4. The third term of (3.12) may be majorized using the fact that $\xi \in C_0^1$, and Lemma 3.3, as follows

$$\begin{aligned} &\left(\sum |K_\delta(z - z_i')|\,|B_i|\right) \max_i |\xi(z_i') - \xi(z_i'')| \\ &\qquad = O(1) \sup_{0 \leq t \leq \alpha_0} ||\xi||_{1,0} \max |z_i' - z_i''| = O(\log \Delta t)(\Delta t). \end{aligned}$$

Using the above, we obtain (3.10a) from (3.12).

4. Accuracy of the Scheme—Convergence to the Solution. We may then compare the positions of the $\tilde{z}_i$ at discrete time steps $n\Delta t$ to the actual positions $z_i(n\Delta t)$ in the following:

THEOREM 4.1. *Let z_i be chosen arbitrarily in B_i at $t = 0$. Then there exists a time interval $[0, \alpha_0]$ in which the following holds: Calculate the motion of the $\tilde{z}_i(n\Delta t)$ at $t = n\Delta t$ for $n = 0, 1, \cdots, [\alpha_0/\Delta t]$ by (2.7). We then have, for each $\epsilon > 0$,*

(4.1) $$\tilde{z}_i(n\Delta t) = z_i(n\Delta t) + O(\Delta t)^{2-\alpha-\epsilon}$$

for $0 \leq n\Delta t \leq \alpha/C_0 = \alpha_0$ and Δt sufficiently small. The constant C_0 is determined by the flow.

Proof. Let $\tilde{z}_i(n\Delta t) = \tilde{z}_i^n = (\tilde{x}_i^n, \tilde{y}_i^n)$ and let $z_i(n\Delta t) = (x_i^n, y_i^n)$. We will prove (4.1) for the x component of z by induction on n. We have

$$x_i^{n+1} - x_i^n = \int_{n\Delta t}^{(n+1)\Delta t} u\, ds = u(z_i^n, n\Delta t)\Delta t + O(\Delta t)^2 \tag{4.2}$$
$$= \Delta t \left(\sum_j K_\delta(z_i^n - z_j^n)k_j \right) + O((\Delta t) \log \Delta t)$$

using Theorem 3.1. Then from (2.7) and (4.2), we have

$$(\tilde{x}_i^{n+1} - x_i^{n+1}) - (\tilde{x}_i^n - x_i^n)$$
$$= \Delta t\left(\sum_j (K_\delta(\tilde{z}_i^n - \tilde{z}_j^n) - K_\delta(z_i^n - z_j^n))k_j \right) + O(\Delta t \log \Delta t) \max |\tilde{z}_i^n - z_i^n|$$
$$= \Delta t(\sum (K_\delta(\tilde{z}_i^n - \tilde{z}_j^n) - K_\delta(\tilde{z}_i^n - z_j^n))k_j) + \Delta t\left(\sum_j (K_\delta(\tilde{z}_i^n - z_j^n) - K_\delta(z_i^n - z_j^n)) \right)k_j$$
$$+ O(\Delta t \log \Delta t) \max_i |\tilde{z}_i^n - z_i^n|.$$

Using Lemma 3.4, we have

$$(\tilde{x}_i^{n+1} - x_i^{n+1}) - (\tilde{x}_i^n - x_i^n) = O(\Delta t) \log \Delta t \max_i |\tilde{z}_i^n - z_i^n|. \tag{4.3}$$

We wish to show by induction that the total error $\max_i |\tilde{z}_i^n - z_i^n|$ at time $n\Delta t$ is of the order

$$(\Delta t^2 |\log \Delta t|)(\Delta t |\log \Delta t| + 1)^{n-1}. \tag{4.4}$$

To see this for $n = 1$, we subtract (2.7) for $n = 0$ from (4.2) for $n = 0$. Since $\tilde{z}_j^0 = z_j^0$ for each j, the right-hand sides of (2.7) and (4.2), both for $n = 0$, are the same except for the term $O(\Delta t^2 \log \Delta t)$ coming from (4.2) for $n = 0$. Hence, for each i, $\tilde{x}_i^1 - x_i^1 = \tilde{x}_i^1 - x_i^1 - (\tilde{x}_i^0 - x_i^0) = \tilde{x}_i^1 - \tilde{x}_i^0 - (x_i^1 - x_i^0)$ is of order $O(\Delta t^2 \log \Delta t)$, which is (4.4) for $n = 1$. Then to find the error at $t = (n + 1)\Delta t$, we use (4.3) to estimate

$$|\tilde{x}_i^{n+1} - x_i^{n+1}| \leqq |(\tilde{x}_i^{n+1} - x_i^{n+1}) - (\tilde{x}_i^n - x_i^n)| + |\tilde{x}_i^n - x_i^n|$$
$$= O(\Delta t \log \Delta t) \max |\tilde{z}_i^n - z_i^n| + O(\max |\tilde{z}_i^n - z_i^n|).$$

Then using our hypothesis of induction (4.4) in the above, we estimate

$$|\tilde{x}_i^{n+1} - x_i^{n+1}| = O(\Delta t \log \Delta t)\cdot O((\Delta t^2 \log \Delta t)(\Delta t |\log \Delta t| + 1)^{n-1})$$
$$+ O((\Delta t^2 \log \Delta t)((\Delta t) |\log \Delta t| + 1)^{n-1})$$
$$= O((\Delta t^2 \log \Delta t)(\Delta t |\log \Delta t| + 1)^n),$$

which can be verified using the binomial theorem. Then using the inequality $1 + x \leqq e^x = \exp(x)$ for $x = O(\Delta t \log \Delta t)$, we have

$$|\tilde{x}_i^{n+1} - x_i^{n+1}| = O((\Delta t^2 \log \Delta t) \exp (O(\Delta t \log \Delta t)n))$$
$$= O(\Delta t^2 \log \Delta t(\Delta t)^{-C_0 n\Delta t})$$

for some constant C_0. So if $C_0 n\Delta t \leqq \alpha$,

$$|\tilde{x}_i^{n+1} - x_i^{n+1}| = O(\Delta t^{2-\alpha}) \log \Delta t = O(\Delta t)^{2-\alpha-\epsilon} \quad \text{for each} \quad \epsilon > 0,$$

which is (4.1).

We may then show how accurately the velocity field may be approximated under the above conditions:

THEOREM 4.2. *Let z_i be points in B_i as before. Let $\tilde{z}_i(n\Delta t)$ be defined by* (2.7). *Define an approximate velocity field $\tilde{u} = (\tilde{u}, \tilde{v})$ by the formulas*

$$\tilde{u}(z, t) = \sum K_\delta(z - z_i^n)k_i, \qquad \tilde{v}(z, t) = \sum L_\delta(z - z_i^n)k_i$$

where $n = [t/\Delta t]$ for $0 \leqq t \leqq \alpha_0$, and α_0 is given by Theorem 4.1. *We then have*

$$\tilde{u}(z, t) - u(z, t) = O(\Delta t)^{2-\alpha-\epsilon} \tag{4.5}$$

for $\epsilon > 0$, all z and $0 \leqq t \leqq \alpha$.

Proof. Since $u \in C^2$, and since $\xi \in C_0^1$, we have (4.5) for sufficiently large z using the fact that, for $z \notin \operatorname{supp} \xi$,

$$|u(z)| \leqq (d(z, \operatorname{supp} \xi))^{-1}(2\pi)^{-1} \iint |\xi|$$

and

$$\tilde{u}(z) = \sum K_\delta(z - z_i^n)k_i \leqq (\tfrac{1}{2}d(z, \operatorname{supp} \xi))^{-1}(2\pi)^{-1} \sum |k_i|$$
$$\leqq (\tfrac{1}{2}d(z, \operatorname{supp} \xi))^{-1}(2\pi)^{-1} \operatorname{supp} |\xi| \sum |B_i|.$$

For z near supp ξ, we may write $u(z, n\Delta t) - \tilde{u}(z, n\Delta t)$ as a difference equal to the difference in the second equation of (3.12). Estimating as we did in the proof of Theorem 3.1, we will get

$$u(z, n\Delta t) - \tilde{u}(z, n\Delta t) = O(\log \Delta t)O(\Delta t)^{2-\alpha-\epsilon} = O(\Delta t)^{2-\alpha-2\epsilon}.$$

Then since ϵ is arbitrary, we have (4.5).

5. Discussion and Generalizations. Our convergence proof gives theoretical evidence that Chorin's new vortex method is correct. The efficacy of the vorticity method lies, in our opinion, in its use of the known particular structure of solutions to the Euler equations. It is analogous to the building of solutions to local approximations as in Glimm [8] for nonlinear hyperbolic equations.

The approximation discussed has practical significance. In many fluid dynamical applications, the region of vorticity is very small compared with the total area of the flow so the total number of vortices is small. This applies in the case of the vorticity generated in the wake of an obstacle. In difference scheme methods (Krzhivitski and Ladyzhenskaya [9], Chorin [3]), the entire velocity field must be approximated. In our method, there is also no problem of boundary conditions at infinity. Finally, the graphical representation of the vortices themselves give excellent qualitative as well as quantitative results. See Chorin [6].

In the case of fluid flow in two dimensions, the solutions to the Euler equations approximate solutions to the Navier-Stokes equations with small viscosity. However, in plasma physics, the Euler equations are precisely the equations of motion, as Levy and Hockney [10] have demonstrated.

For the Navier-Stokes equations with boundary we conjecture our scheme and the more general scheme may be proved to converge in the large and we expect soon to publish results in that direction.

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