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Symmetrization of the Fluid Dynamic Matrices with Applications

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Abstract. The matrices occurring in the equations of inviscid fluid dynamics are simultaneously symmetrized by a similarity transformation. The resulting matrices decompose into several lower-dimensional blocks. In addition these blocks are more sparse than previously obtained. These properties are then used to find a sufficiency proof for an improved version of the two-step Richtmyer method.

1. Symmetrization of the Fluid Dynamic Matrices. The linear stability analysis for many numerical schemes for hyperbolic partial differential equations assumes that the equations are symmetric or, equivalently, that they are simultaneously symmetrizable. Although the fluid equations are nonlinear, in practice one uses the linear stability criterion for lack of any other choice. Furthermore, Strang [10] has shown that for sufficiently smooth flows the linear theory is valid for the nonlinear equations. It is well known that the equations of inviscid fluid dynamics form a symmetric hyperbolic system when the dependent variables are chosen as pressure, components of velocity, and entropy. Furthermore, Gudonov [4] has shown that it is possible to write the equations in conservation form in such a manner that the corresponding semilinear vector equations form a symmetric hyperbolic system. (For general connections between conservation laws and symmetry, see Friedrichs and Lax [3].)

In this paper, we shall show that the matrices with dependent variables, density, momentum components, and energy can be simultaneously symmetrized. By the Kreiss matrix theorem [8], the stability of the difference equations is not affected by a similarity transform. It therefore follows that all proofs of linear stability for symmetric systems also apply to the fluid equations with the physically conserved quantities as dependent variables. It should be noted that, to correctly predict shock speeds, one must use the equations with ρ, ρu, ρv and E as dependent variables (see for example [5], [7]). Furthermore, the symmetrized matrices are sparse which can be utilized to simplify stability analyses. In fact, the matrices contain more zeros than one would ordinarily obtain using the available parameters in the similarity transformation. In many cases, it is advantageous to introduce new independent variables to simplify the treatment of boundaries. This will introduce new matrices which are functions of the original matrices. Use of the sparse transformed matrices will simplify the evaluation of the von Neumann condition for the new matrices. In later sections, we shall use this sparseness to derive a sufficient condition for special schemes.

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For simplicity, we shall discuss the equations in two space dimensions and for a polytropic gas. Three space dimensions or more general equations of state do not present any major difficulties. In two space dimensions, the inviscid fluid dynamic equations can be written

$$w_t + f_x + g_y = 0 \tag{1}$$

where

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \qquad E = \frac{p}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2),$$

$$f = \begin{pmatrix} m \\ -\frac{\gamma - 3}{2}\frac{m^2}{\rho} + (\gamma - 1)\left(E - \frac{n^2}{2\rho}\right) \\ \frac{mn}{\rho} \\ \frac{1-\gamma}{2\rho^2} m(m^2 + n^2) + \frac{\gamma m E}{\rho} \end{pmatrix}, \quad g = \begin{pmatrix} m \\ \frac{mn}{\rho} \\ -\frac{\gamma - 3}{2}\frac{n^2}{\rho} + (\gamma - 1)\left(E - \frac{m^2}{\rho}\right) \\ \frac{1-\gamma}{2\rho^2} n(m^2 + n^2) + \frac{\gamma n E}{\rho} \end{pmatrix}.$$

Equivalently, Eq. (1) can be written as

$$w_t + Aw_x + Bw_y = 0, \tag{2}$$

$$A = -\begin{pmatrix} 0 & -1 & 0 & 0 \\ \frac{3-\gamma}{2}u^2 + \frac{1-\gamma}{2}v^2 & (\gamma - 3)u & (\gamma - 1)v & 1 - \gamma \\ uv & -v & -u & 0 \\ \frac{\gamma E u}{\rho} + (1-\gamma)u(u^2 + v^2) & -\frac{\gamma E}{\rho} + \frac{\gamma - 1}{2}(3u^2 + v^2) & (\gamma - 1)uv & -\gamma u \end{pmatrix},$$

$$B = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ uv & -v & -u & 0 \\ \frac{3-\gamma}{2}v^2 + \frac{1-\gamma}{2}u^2 & (\gamma - 1)u & (\gamma - 3)v & 1 - \gamma \\ \frac{\gamma u E}{\rho} + (1-\gamma)v(u^2 + v^2) & (\gamma - 1)uv & \frac{-\gamma E}{\rho} + \frac{\gamma - 1}{2}(3v^2 + u^2) & -\gamma v \end{pmatrix}.$$

Let

$$(3)\qquad T = \begin{pmatrix} \frac{\gamma-1}{2\rho c}(u^2+v^2) & -\frac{(\gamma-1)u}{\rho c} & -\frac{(\gamma-1)v}{\rho c} & \frac{\gamma-1}{\rho c} \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{-c^2}{p_s} + \frac{\gamma-1}{2p_s}(u^2+v^2) & -\frac{(\gamma-1)u}{p_s} & -\frac{(\gamma-1)v}{p_s} & \frac{\gamma-1}{p_s} \end{pmatrix}$$

where $c^2 = \gamma p/\rho$, c is the sound speed. $p_s = p/c_v$. We can then calculate the inverse of T.

$$T^{-1} = \begin{pmatrix} \frac{\rho}{c} & 0 & 0 & \frac{-p_s}{c^2} \\ \frac{\rho u}{c} & \rho & 0 & \frac{-up_s}{c^2} \\ \frac{\rho v}{c} & 0 & \rho & \frac{-vp_s}{c^2} \\ \frac{\rho c}{\gamma-1} + \frac{\rho}{2c}(u^2+v^2) & \rho u & \rho v & \frac{-p_s}{2c^2}(u^2+v^2) \end{pmatrix}.$$

It then follows that

$$(4)\qquad A_0 = TAT^{-1} = \begin{pmatrix} u & c & 0 & 0 \\ c & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}, \qquad B_0 = TBT^{-1} = \begin{pmatrix} v & 0 & c & 0 \\ 0 & v & 0 & 0 \\ c & 0 & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix}.$$

It is also possible to diagonalize one of these matrices. For example, let

$$T_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A_1 = T_1 A_0 T_1^{-1} = \begin{pmatrix} u+c & 0 & 0 & 0 \\ 0 & u-c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix},$$

$$B_1 = T_1 B_0 T_1^{-1} = \begin{pmatrix} v & 0 & \frac{\sqrt{2}}{2}c & 0 \\ 0 & v & \frac{-\sqrt{2}}{2}c & 0 \\ \frac{\sqrt{2}}{2}c & \frac{-\sqrt{2}}{2}c & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix}.$$

We see that B_1 still contains zeros (i.e., elements b_{12}, b_{21}) which would not ordinarily occur. So an extra measure of spareness appears after the similarity transformation. This is in addition to the effective reduction in dimension due to the common eigenvector of the matrices A and B.

2. A Sufficient Stability Criterion. The matrices A_0 and B_0 have other properties besides that of symmetry. It is immediately apparent that the stability criterion for any finite difference scheme depends only on the values of u, v, and c and not on any other dependent variables. This, of course, is obvious from a physical point of view. Also, the transformed amplification matrix can be written as the direct sum of a matrix of rank three together with a matrix of rank one, immediately reducing the dimensionality of the problem (a corresponding statement holds for three space dimensions).

We denote by G the amplification matrix of the finite difference scheme, while G_0 denotes the similarity transform, by T, of G. In order to facilitate the following analysis, we shall introduce a norm equivalent to the original L^2 norm. Let $(x, y)_T = (Tx, Ty) = (T^*Tx, y)$ and let $||\cdot||_T$ be the matrix norm induced by the vector norm. This quadratic form is positive definite since T is invertible and, hence, has no zero eigenvalues. Furthermore,

$$\frac{1}{K}\,||\cdot|| \leqq ||\cdot||_T \leqq K\,||\cdot|| \quad \text{with } K = \frac{\max|\text{e.v.}(T^*T)|}{\min|\text{e.v.}(T^*T)|}$$

and so the two norms are equivalent.

When we are able to show that the norm of G_0 is less than one, we have strong stability for G in an equivalent norm since $||G_0|| = ||G||_T$. Hence, by condition (H) of the Kreiss matrix theorem, the difference scheme is stable. Because of the special form of the matrices A_0 and B_0, we are able to obtain a simplified condition which is equivalent to G_0 being strongly stable. We thus can obtain a sufficient condition for stability. $||G_0||^2 = ||G_0^*G_0||$ = spectral radius $(G_0^*G_0)$. So, we must find a condition that will guarantee that the largest eigenvalue of $G_0^*G_0$ is less than or equal to one (the eigenvalues are real since $G_0^*G_0$ is Hermitian).

From the form of A_0 and B_0, it follows that

$$G_0^*G_0 = I + \tilde{G} = I + \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 \\ \bar{g}_{12} & g_{22} & g_{23} & 0 \\ \bar{g}_{13} & \bar{g}_{23} & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix} \tag{5}$$

where g_{ij} are complex functions of ξ, η, the dual Fourier variables. For stability, it is thus sufficient that $\tilde{G}$ be negative semidefinite, i.e., that $\tilde{G}$ has no positive eigenvalues.

(6a) $$0 = \det(\tilde{G} - \mu I) = (g_{44} - \mu)(-\mu^3 + a_2\mu^2 + a_1\mu + a_0).$$

We shall assume that $\mu = g_{44}$ is negative since this is equivalent to stability for a single scalar equation. Any difference method which is unstable for this simple system is not to be considered. We shall thus confine our attention to the cubic equation

(6b) $$f(\xi, \eta; \mu) = -\mu^3 + a_2\mu^2 + a_1\mu + a_0 = 0,$$

where

$$a_2 = g_{11} + g_{22} + g_{33},$$

(7) $$a_1 = -(g_{11}g_{22} + g_{11}g_{33} + g_{22}g_{33} - |g_{12}|^2 - |g_{13}|^2 - |g_{23}|^2),$$

$$a_0 = g_{11}g_{22}g_{33} + 2 \operatorname{Re}(g_{12}g_{13}g_{23}) - g_{11}|g_{23}|^2 - g_{22}|g_{13}|^2 - g_{33}|g_{12}|^2.$$

A necessary and sufficient condition for the roots of f to be nonpositive is that

(8) $$a_0 \leqq 0, \qquad a_1 \leqq 0, \qquad a_2 \leqq 0.$$

This is true since the a_i are real and the roots are known to be real, since G^*G is Hermitian. In fact, if the roots are all simple, it is enough to find the conditions under which a_0 changes from negative to positive in order to find the stability criterion. In this case, we have only one trigonometric inequality to analyze, rather than a matrix inequality. When a double root can occur, it is necessary to verify the inequalities for both a_1 and a_0. It is not necessary to check the inequality for a_2 since, at a triple zero, a_0 will become zero and so it suffices to verify the inequalities (8) for a_1 and a_0. Even in cases where the stability criterion cannot be given explicitly, it is easier to sample two scalar implicit conditions rather than sample the eigenvalues of a four by four matrix over a range of the Fourier variables. In addition, if one can solve the trigonometric equality (rather than inequality), then it is sufficient to verify the conditions under which $a_0 = 0$ for some ξ, η, $0 < \xi < \pi$, $0 < \eta < \pi$, since this must occur at a zero root of (6b). Assuming that the difference equation is not unconditionally unstable, there is a range of u, v, c for which the roots of (6b) are negative and so, the only way the scheme can become unstable, is for $f(\xi, \eta; \mu)$ to have a zero root. Therefore, instability sets in only at those values of u, v, and c for which $a_0 = 0$.

In the following section, we shall explicitly calculate the a_i for a large class of second-order difference methods. For one specific scheme, we shall explicitly construct conditions on u, v, c that guarantee that $a_i \leqq 0$ and so we have an explicit sufficient condition for stability. For this particular scheme, the condition is necessary as well as sufficient.

3. Applications. In this section, we shall determine the trigonometric inequalities that result from Eq. (8) for a class of finite difference equations. We shall then solve these inequalities for a particular scheme to determine a sufficient condition for stability. The class of difference equations that we consider is a generalization of the Lax-Wendroff method [7]. This generalization includes the two-step schemes of Richtmyer [8] and Burstein [1].

We consider finite difference schemes which can be characterized by an amplification matrix of the form

$$
\begin{aligned}
G = I &- \lambda^2[\alpha^2(\xi, \eta)A^2 + \beta^2(\xi, \eta)B^2 + \gamma(\xi, \eta)(AB + BA)] \\
&+ i\lambda[\delta(\xi, \eta)A + \epsilon(\xi, \eta)B]
\end{aligned} \tag{9}
$$

where $\lambda = \Delta t/\Delta$, $\Delta = \Delta x = \Delta y$; ξ, η are the Fourier variables. For this amplification matrix, the g_{ij} (see Eq. (5)) are

$$
\begin{aligned}
g_{11} = {} & -(2\alpha^2 - \delta^2)(u^2 + c^2) - (2\beta^2 - \epsilon^2)(v^2 + c^2) - 2(2\gamma - \delta\epsilon)uv \\
& + \alpha^4(u^4 + 6u^2c^2 + c^4) + \beta^4(v^4 + 6v^2c^2 + c^4) + 4\gamma^2(u^2v^2 + c^2u^2 + c^2v^2) \\
& + 2\alpha^2\beta^2(u^2v^2 + c^2u^2 + c^2v^2 + c^4) + 4\alpha^2\gamma(u^3v + 3uvc^2) \\
& + 4\beta^2\gamma(uv^3 + 3uvc^2), \\
g_{22} = {} & -(2\alpha^2 - \delta^2)(u^2 + c^2) - (2\beta^2 - \epsilon^2)v^2 - 2(2\gamma - \delta\epsilon)uv \\
& + \alpha^4(u^4 + 6u^2c^2 + c^4) + \beta^4v^4 + \gamma^2(4u^2v^2 + 4v^2c^2 + c^4) \\
& + 2\alpha^2\beta^2(u^2v^2 + v^2c^2) + 4\alpha^2\gamma(u^3v + 3uvc^2) + 4\beta^2\gamma uv^3, \\
g_{33} = {} & -(2\alpha^2 - \delta^2)u^2 - (2\beta^2 - \epsilon^2)(v^2 + c^2) - 2(2\gamma - \delta\epsilon)uv \\
& + \alpha^4u^4 + \beta^4(v^4 + 6v^2c^2 + c^4) + \gamma^2(4u^2v^2 + 4u^2c^2 + c^4) \\
& + 2\alpha^2\beta^2(u^2v^2 + u^2c^2) + 4\alpha^2\gamma u^3v + 4\beta^2\gamma(uv^3 + 3uvc^2), \\
g_{12} = {} & -2(2\alpha^2 - \delta^2)uc - 2(2\gamma - \delta\epsilon)vc + 4\alpha^4(u^3c + uc^3) \\
& + 2\gamma^2(4uv^2 + uc^3) + 2\alpha^2\beta^2(2uv^2c + uc^3) + 4\alpha^2\gamma(3u^2vc + vc^3) \\
& + 4\beta^2\gamma(v^3c + vc^3) - i(\beta^2\delta - \gamma\epsilon)c^3, \\
g_{13} = {} & -2(2\beta^2 - \epsilon^2)vc - 2(2\gamma - \delta\epsilon)uc + 4\beta^4(v^3c + vc^3) \\
& + 2\gamma^2(4u^2v + vc^3) + 2\alpha^2\beta^2(2u^2vc + vc^3) + 4\alpha^2\gamma(u^3c + uc^3) \\
& + 4\beta^2\gamma(3uv^2c + uc^3) + i(\gamma\delta - \alpha^2\epsilon)c^3, \\
g_{23} = {} & -(2\gamma - \delta\epsilon)c^2 + 8\gamma^2uvc^2 + 4\alpha^2\beta^2uvc^2 + \alpha^2\gamma(6u^2c^2 + c^4) \\
& + \beta^2\gamma(6v^2c^2 + c^4).
\end{aligned} \tag{10}
$$

Now we consider a modified version of the Richtmyer two-step scheme. This scheme has the advantage over the original Richtmyer scheme [8] that its domain of dependence is a square rather than a triangle; this is of great importance near boundaries. Furthermore, there is no splitting between even and odd mesh points and so increased stability for long term integrations. In addition, Eilon, Gottlieb and Zwas [2] have shown that this modified Richtmyer scheme is computationally faster than the space splitting schemes proposed by Strang [9]. This scheme is also more efficient than a similar scheme proposed by Burstein [1] and an explicit stability condition can be formulated for the inviscid fluid dynamic equations while none has thus far been found for the Burstein equations. Further comparisons between these schemes will be discussed in a future paper.

We consider the general equation

(11) $$u_t = f_x(u) + g_y(u).$$

Let

$$\hat{u}_{i+1/2,j+1/2} = \tfrac{1}{4}(u^n_{i+1,j+1} + u^n_{i+1,j} + u^n_{i,j+1} + u^n_{i,j})$$
$$+ \tfrac{1}{4}\lambda(f^n_{i+1,j+1} - f^n_{i,j+1} + f^n_{i+1,j} - f^n_{i,j})$$

(12) $$+ \tfrac{1}{4}\lambda(g^n_{i+1,j+1} - g^n_{i+1,j} + g^n_{i,j+1} - g^n_{i,j}),$$

$$u^{n+1}_{i,j} = u^n_{i,j} + \tfrac{1}{2}\lambda(\hat{f}_{i+1/2,j+1/2} - \hat{f}_{i-1/2,j+1/2} + \hat{f}_{i+1/2,j-1/2} - \hat{f}_{i-1/2,j-1/2})$$
$$+ \tfrac{1}{2}\lambda(\hat{g}_{i+1/2,j+1/2} - \hat{g}_{i+1/2,j-1/2} + \hat{g}_{i-1/2,j+1/2} - \hat{g}_{i-1/2,j-1/2})$$

where $\lambda = \Delta t/\Delta$, $\Delta = \Delta x = \Delta y$, $\hat{f} = f(\hat{u})$. For this particular scheme, the g_{ij} are real. Let

(13)
$$S(\xi, \eta) = \frac{1}{2} + \frac{1}{4}\frac{(1 - \cos \xi)(1 - \cos \eta)}{1 - \cos \xi \cos \eta} \geqq \frac{1}{2},$$
$$m = \left(\frac{\Delta t}{\Delta}\right)^2 \frac{\alpha u^2 + \beta v^2 + 2\gamma uv}{\alpha + \beta} = \left(\frac{\Delta t}{\Delta}\right)^2 \left(\left(\frac{\alpha}{\alpha + \beta}\right)^{1/2} u + \left(\frac{\beta}{\alpha + \beta}\right)^{1/2} v\right)^2,$$

and $K_1 = 4cm^{1/2}(-S + m + c^2)$; $K_2 = m(-2S + m) + 2(-S + 3m)c^2 + c^4$. It then follows from Eq. (7) that

(14) $$a_0 = (\alpha + \beta)^6(-2S + m)m(K_2^2 - K_1^2),$$

but

$$K_1 + K_2 = (m^{1/2} + c)^2(-2S + (m^{1/2} + c)^2),$$
$$K_2 - K_1 = (m^{1/2} - c)^2(-2S + (m^{1/2} - c)^2).$$

Therefore, if $(m^{1/2} + c)^2 \leqq 2S$, it follows that $K_1 + K_2 \leqq 0$, $K_2 - K_1 \leqq 0$, and so $K_2 \leqq 0$, also $K_1 \leqq -K_2 = |K_2|$, $-K_1 \leqq -K_2 = |K_2|$. Therefore, $|K_1| \leqq |K_2|$ which implies that $a_0 \leqq 0$. Thus, the requirement that a_0 be less than or equal to zero is equivalent to

(15) $$\frac{(\Delta t)^2}{\Delta}(m^{1/2} + c)^2 \leqq 2 \min_{\xi,\eta} S(\xi, \eta) = 1,$$

but

$$m = (\alpha u^2 + \beta v^2 + 2\gamma uv)/(\alpha + \beta)$$
$$= u^2 + v^2 - \left(\left(\frac{\alpha}{\alpha + \beta} - 1\right)^{1/2} u - \left(\frac{\beta}{\alpha + \beta} - 1\right)^{1/2} v\right)^2 \leqq u^2 + v^2$$

with equality for $\alpha = u/(u^2 + v^2)^{1/2}$, $\beta = v/(u^2 + v^2)^{1/2}$. So we require that

(16) $$(\Delta t/\Delta)[(u^2 + v^2)^{1/2} + c] \leqq 1.$$

Similarly,

(17) $$a_1 = (\alpha + \beta)^6(K_1^2 - (K_2 + 2m(-2S + m))K_2)$$

as before $(m^{1/2} + c)^2 \leqq 1$ implies that $K_2 \leqq 0$ and $2m(-2S + m) \leqq 0$ and hence that

$$a_1 \leqq (\alpha + \beta)^6 (K_1^2 - K_2^2) \leqq 0. \tag{18}$$

We have thus shown that if $(\Delta t/\Delta)[(u^2 + v^2)^{1/2} + c] \leqq 1$, then both a_0 and a_1 are nonpositive and, hence, that the difference scheme (12) is stable, under these conditions, for the inviscid fluid dynamic equations. Zwas [11] has shown that this condition is also necessary and so the condition is the best possible. Similarly, the sparse symmetric form of the fluid dynamic matrices (Eq. (4)) can be used to simplify the stability analysis, both analytic and numerical, of other difference methods.

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1. S. Z. Burstein, "Finite difference calculations for hydrodynamic flows containing discontinuities," *J. Computational Phys.*, v. 1, 1966, pp. 198–222.

2. B. Eilon, D. Gottlieb & G. Zwas, "Numerical stabilizers and computing time for second order accurate schemes," *J. Computational Phys.*, v. 9, 1972, pp. 387–397.

3. K. O. Friedrichs & P. D. Lax, "Systems of conservation equations with a convex extension," *Proc. Nat. Acad. Sci. U.S.A.*, v. 68, 1971, pp. 1686–1688. MR **44** #3016.

4. S. K. Godunov, "The problem of a generalized solution in the theory of quasi-linear equations and in gas dynamics," *Uspehi Mat. Nauk*, v. 17, 1962, no. 3 (105), pp. 147–158 = *Russian Math. Surveys,* v. 17, 1962, no. 3, pp. 145–156. MR **27** #5445.

5. A. Kasahara & D. Houghton, "An example of nonunique discontinuous solutions in fluid dynamics," *J. Computational Phys.*, v. 4, 1969, pp. 377–388.

6. P. D. Lax & B. Wendroff, "Systems of conservation laws," *Comm. Pure Appl. Math.*, v. 13, 1960, pp. 217–237. MR **22** #11523.

7. P. D. Lax & B. Wendroff, "Difference schemes for hyperbolic equations with higher order accuracy," *Comm. Pure Appl. Math.*, v. 17, 1964, pp. 381–398. MR **30** #722.

8. R. D. Richtmyer & K. W. Morton, *Difference Methods for Initial Value Problems*, 2nd ed., Interscience, New York, 1967. MR **36** #3515.

9. G. W. Strang, "Accurate partial difference methods. II: Nonlinear problems," *Numer. Math.*, v. 6, 1964, pp. 37–46. MR **29** #4215.

10. G. W. Strang, "On the construction and comparison of difference schemes, "*SIAM J. Numer. Anal.*, v. 5, 1968, pp. 506–517. MR **38** #4057.

11. G. Zwas, "On two step Lax-Wendroff methods in several dimensions," *Numer. Math.* (To appear.)