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Approximate Solution of the Differential Equation $y''=f(x, y)$ with Spline Functions

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Abstract. An approximate spline is constructed for the solution of Cauchy's problem regarding a second-order differential equation. The existence, uniqueness and convergence of the approximate spline solution are investigated.

1. Introduction. Let $(\mathfrak{S}_m, C^k)$ be the class of spline functions with respect to the set of knots $\{x_i\}$. This class consists of piecewise-polynomial functions of degree m, smoothly connected in the knots, up to the derivatives of order k $(k < m)$.

We shall use spline functions of class $(\mathfrak{S}_m, C^{m-1})$ in approximating the solution of the Cauchy problem for $y'' = f(x, y)$.

F. R. Loscalzo and T. D. Talbot ([3], [4]) made use of spline functions in approximating solution of the Cauchy problem for $y' = f(x, y)$. In [6], Manabu Sakai approximated the solutions of two-point boundary value problems for the second-order equations by spline functions. Recently [5], the author studied the approximation of solutions of systems of differential equations by spline functions.

For our purpose, we shall need consistency relations which hold for any spline functions of $(\mathfrak{S}_m, C^{m-1})$ with equidistant knots $x_k = kh (k = 1, \cdots, n-1)$. We have

THEOREM 1. *For any spline function* $\mathfrak{s} \in (\mathfrak{S}_m, C^{m-1})$, $m \geqq 3$, *there are linear relations between the quantities* $\mathfrak{s}(kh)$, $\mathfrak{s}'(kh)$; $\mathfrak{s}(kh)$, $\mathfrak{s}''(kh)$, $k = 0, \cdots, m-1$, *given by*

$$\sum_{k=0}^{m-1} a_k^{(m)} \mathfrak{s}(kh) = h \sum_{k=0}^{m-1} b_k^{(m)} \mathfrak{s}'(kh), \tag{1}$$

$$\sum_{k=0}^{m-1} c_k^{(m)} \mathfrak{s}(kh) = h^2 \sum_{k=0}^{m-1} b_k^{(m)} \mathfrak{s}''(kh) \tag{2}$$

with the coefficients

$$a_k^{(m)} = (m-1)!\,[Q_m(k) - Q_m(k+1)], \tag{3}$$

$$c_k^{(m)} = (m-1)!\,[Q_{m-1}(k+1) - 2Q_{m-1}(k) + Q_{m-1}(k-1)], \tag{4}$$

$$b_k^{(m)} = (m-1)!\,Q_{m+1}(k+1), \tag{5}$$

where

$$Q_{m+1}(x) = \frac{1}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} (x-i)_+^m$$

is a B-spline.

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More details on this theorem may be found in [6], [3], [4], [8].

2. Construction of Approximate Spline Solution. Consider

(6) $$y'' = f(x, y)$$

where f: $[0, B] \times \mathbf{R} \to \mathbf{R}$ is a sufficiently smooth function. We attach to Eq. (6) the Cauchy conditions

(7) $$y(0) = y_0 , \qquad y'(0) = y_0' .$$

Suppose the function f satisfies a Lipschitz condition with constant A:

(8) $$|f(x, y) - f(x, Y)| \leqq A|y - Y|, \qquad \forall\ (x, y), (x, Y) \in [0, B] \times \mathbf{R}.$$

Under these conditions there exists a unique solution y of (6)–(7). Let $[0, b]$ be its domain.

Following the idea of [3], we construct a polynomial spline function of degree m ($m \geqq 3$) to approximate the exact solution y of (6)–(7).

Let $n > m$ be an integer, $h = b/n$ and $\mathfrak{s}$: $[0, b] \to \mathbf{R}$ the spline function of degree m and class C^{m-1} with knots $x = h, 2h, \cdots, (n - 1)h$. The first component of $\mathfrak{s}$ on $[0, h]$ is

(9) $$\mathfrak{s}(x) = y(0) + y'(0)x + \cdots + \frac{y^{(m-1)}(0)}{(m-1)!} x^{m-1} + \frac{a_0}{m!} x^m, \qquad 0 \leqq x \leqq h,$$

where the coefficient a_0 is as yet undetermined. We determine a_0 by requiring that $\mathfrak{s}$ satisfy (6) in $x = h$. This gives us

$$\mathfrak{s}''(h) = f(h, \mathfrak{s}(h))$$

which determines a_0. Now, if the polynomial (9) is determined, define the spline function $\mathfrak{s}$ on the next interval $[h, 2h]$ by

$$\mathfrak{s}(x) = \sum_{j=0}^{m-1} \frac{\mathfrak{s}^{(j)}(h)}{j!} (x - h)^j + \frac{a_1}{m!} (x - h)^m, \qquad h \leqq x \leqq 2h,$$

where a_1 will be determined such that $\mathfrak{s}$ satisfies Eq. (6) in $x = 2h$, i.e., $\mathfrak{s}''(2h) = f(2h, \mathfrak{s}(2h))$.

Continuing in this way, we obtain a spline function satisfying

$$\mathfrak{s}''(kh) = f(kh, \mathfrak{s}(kh)), \qquad k = 0, \cdots, n.$$

THEOREM 2. *If $h < (m(m-1)/A)^{1/2}$ then the spline function $\mathfrak{s}$ given by the above construction exists and is unique.*

Proof. On the interval $[kh, (k + 1)h]$ we define

(10) $$\mathfrak{s}(x) = \sum_{j=0}^{m-1} \frac{\mathfrak{s}^{(j)}(kh)}{j!} (x - kh)^j + \frac{a_k}{m!} (x - kh)^m \equiv A_k(x) + \frac{a_k}{m!} (x - kh)^m,$$
$$x \in [kh, (k + 1)h], \qquad k = 0, \cdots, n - 1.$$

$A_k(x)$ is known by continuity conditions. Let us prove that a_k may be uniquely determined from

(11) $$\mathfrak{s}''((k + 1)h) = f((k + 1)h, \mathfrak{s}(k + 1)h).$$

Replacing $\mathfrak{s}$ in (11), we get the equation

$$(12)\quad a_k = \frac{(m-a)!}{h^{m-2}}\left\{f\left[(k+1)h,\ A_k((k+1)h)+\frac{h^m}{m!}a_k\right] - A_k''((k+1)h)\right\} = g_k(a_k)$$

for the unknown a_k.

Define $G_k : \mathrm{R} \to \mathrm{R}$ by $a_k \to g_k(a_k)$, $a_k \in \mathrm{R}$. We show that under the conditions of the theorem, operator G_k is a contraction thus having a unique fixed point.

Let a_k^1, $a_k^2 \in \mathrm{R}$, and their distance $\rho(a_k^1, a_k^2) = |a_k^1 - a_k^2|$.

According to the Lipschitz condition (8), it follows that

$$\rho(G_k(a_k^1), G_k(a_k^2)) = |g_k(a_k^1) - g_k(a_k^2)| \leqq \frac{h^2 A}{m(m-1)}\rho(a_k^1, a_k^2).$$

If $h^2A/m(m-1) < 1$, G_k is a contraction operator and Eq. (12) has a unique solution. This completes the proof.

THEOREM 3. *The values* $\mathfrak{s}(jh)$, $j = 0, \cdots, n$, *of the spline function constructed above are precisely the values furnished by the discrete multistep method described by the recurrence relation*

$$(13)\qquad \sum_{j=0}^{m-1} c_j^{(m)} y_{j-m+k+1} = h^2 \sum_{j=0}^{m-1} b_j^{(m)} y''_{j-m+k+1}, \qquad k = m-1, \cdots, n,$$

where coefficients $c_j^{(m)}$, $b_j^{(m)}$ *are given by* (4), (5), *if the starting values*

$$(14)\qquad y_0 = \mathfrak{s}(0), \qquad y_1 = \mathfrak{s}(h), \cdots, y_{m-2} = \mathfrak{s}((m-2)h)$$

are used.

Proof. For $h < (m(m-1)/A)^{1/2}$, only one sequence $\{y_j\}$, $j = m-1, \cdots, n$, satisfies relation (13) with starting values (14). By the consistency relation (2), the sequence $\mathfrak{s}(jh)$, $j = m-1, \cdots, n$, satisfies (13) and obviously has starting value (14).

Thus the values $\mathfrak{s}(jh)$, $j = m-1, \cdots, n$, must coincide with the values y_j, $j = m-1, \cdots, n$, generated by the corresponding multistep method.

Theorem 3 tells us that the approximate spline solution of degree m yields the same values as the discrete method of $(m-1)$-steps on x_k.

In the sequel, we shall be concerned with estimating the error of approximation of the solution of problems (6)–(7) by splines as well as with convergence of the approximation $\mathfrak{s}$ to the exact solution y for $h \to 0$. We now define the step function $\mathfrak{s}^{(m)}$ at the knots $x_k = kh$, $k = 1, \cdots, n-1$ (see [4, p. 437]) by the usual arithmetic mean:

$$(15)\qquad \mathfrak{s}^{(m)}(x_k) = \tfrac{1}{2}[\mathfrak{s}^{(m)}(x_k - \tfrac{1}{2}h) + \mathfrak{s}^{(m)}(x_k + \tfrac{1}{2}h)], \qquad k = 1, \cdots, n-1.$$

LEMMA 1. *If* $|\mathfrak{s}(x_k) - y(x_k)| < Kh^p$ *and* $\mathfrak{s}''(x_k) = f(x_k, \mathfrak{s}(x_k))$ *then there exists a constant* K_2 *such that*

$$|\mathfrak{s}(x_k) - y(x_k)| < K_2 h^p \quad \textit{and} \quad |\mathfrak{s}''(x_k) - y''(x_k)| < K_2 h^p.$$

Proof. Applying Lipschitz condition (8) it follows that

$$|\mathfrak{s}''(x_k) - y''(x_k)| = |f(x_k, \mathfrak{s}(x_k)) - f(x_k, y(x_k))| \leqq A|\mathfrak{s}(x_k) - y(x_k)| < AKh^p.$$

We can take $K_2 = \max\{K, AK\}$.

LEMMA 2 (LOSCALZO-TALBOT [4, p. 438]). *Let* $y \in C^{m+1}[0, b]$, *and let* $\mathfrak{s}$ *be a spline* *function of degree m having its knots at the points* x_k, $k = 1, \cdots, n-1$, *and such that the conditions*

(16) $$|\mathfrak{s}^{(r)}(x_k) - y^{(r)}(x_k)| = O(h^{p_r}), \qquad r = 0, \cdots, m-1, k = 0, \cdots, n-1,$$

(17) $$|\mathfrak{s}^{(m)}(x) - y^{(m)}(x)| = O(h), \qquad x_k < x < x_{k+1}, k = 0, \cdots, n-1$$

are satisfied. Then,

(18) $$|\mathfrak{s}(x) - y(x)| = O(h^p)$$

where

(19) $$p = \min_{r=0,\cdots,m} (r + p_r) \qquad (p_m = 1)$$

and furthermore

(20) $$|\mathfrak{s}^{(m)}(x) - y^{(m)}(x)| = O(h), \qquad x \in [0, b].$$

In what follows we study the approximation of a solution by spline functions of degree $m = 3$ (cubic) and $m = 4$. For brevity we denote $x_k = kh$, $y_k = y(x_k)$, $y'_k = y'(x_k)$, $y''_k = y''(x_k)$ $(k = 0, \cdots, n)$, and analogously for $\mathfrak{s}(x_k)$, $\mathfrak{s}'(x_k)$, $\mathfrak{s}''(x_k)$.

3. Cubic Spline Functions Approximating the Solution. Theorem 1 gives, for $m = 3$,

$$\mathfrak{s}_{k+1} - 2\mathfrak{s}_k + \mathfrak{s}_{k-1} = \tfrac{1}{6}h^2(\mathfrak{s}''_{k+1} + 4\mathfrak{s}''_k + \mathfrak{s}''_{k-1}), \qquad k = 1, \cdots, n-1.$$

By Theorem 3 the cubic spline function yields the same values on the knots as the discrete multistep method based on the recurrence formula

(21) $$\begin{aligned} y_{k+1} - 2y_k + y_{k-1} &= \tfrac{1}{6}h^2(y''_{k+1} + 4y''_k + y''_{k-1}) \\ &= \tfrac{1}{6}h^2[f(x_{k+1}, y_{k+1}) + 4f(x_k, y_k) + f(x_{k-1}, y_{k-1})] \end{aligned}$$

if starting values y_0 and $y_1 = \mathfrak{s}(h)$ are used.

The multistep method (21) has the degree of exactness three, provided that starting values y_0, y_1 have third-order accuracy (see [2, p. 295]).

LEMMA 3. *Let* $m = 3$. *Then there exists a constant* K *such that* $|\mathfrak{s}(h) - y(h)| < Kh^3$:

Proof. From the developments

$$\mathfrak{s}(h) = y_0 + hy'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{6} a_0,$$

$$y(h) = y_0 + hy'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{6} y'''_0 + \frac{h^4}{24} y^{(4)}(\xi), \qquad 0 < \xi < h,$$

we have

(22) $$|\mathfrak{s}(h) - y(h)| = \tfrac{1}{6}h^3 |(a_0 - y'''_0) - \tfrac{1}{4}hy^{(4)}(\xi)|.$$

The proof of the lemma is reduced to showing that a_0 is uniformly bounded as a function of h. From (12), it follows that, for $m = 3$, we have

(23) $$g_0(a_0) = \frac{1}{h}\left[f\left(h, y_0 + hy'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{6} a_0\right) - y''_0\right].$$

The function $g_0(u)$ is a contraction if $h < (6/A)^{1/2}$.
In particular for $h < (1/A)^{1/2}$, we have

$$|g_0(u_1) - g_0(u_2)| < \tfrac{1}{6}|u_1 - u_2|, \qquad u_1, u_2 \in \mathbf{R}.$$

Taking $u_1 = a_0$, $u_2 = 0$, we obtain

$$|g_0(a_0)| - |g_0(0)| \leqq |g_0(a_0) - g_0(0)| < \tfrac{1}{6}|a_0|.$$

But $g_0(a_0) = a_0$, so that $|a_0| - |g_0(0)| < \frac{1}{6}|a_0|$ implies

$$|a_0| < \tfrac{6}{5}|g_0(0)|. \tag{24}$$

From (23), (24), it follows that

$$g_0(0) = \frac{1}{h}\left|f\left(h, y_0 + hy_0' + \frac{h^2}{2}y_0''\right) - y_0''\right| = \frac{1}{h}|y''(h) + O(h^3) - y_0''|$$

$$= \frac{1}{h}|y_0'' + O(h) - y_0''| \leqq M$$

for some constant M. Since uniform spacing is required over the interval [0, b], there is only a finite number of possible values of h between $(1/A)^{1/2}$ and $(6/A)^{1/2}$, so that a_0 is uniformly bounded for all $h < (6/A)^{1/2}$, and the proof of the lemma is completed.

On the basis of Lemma 3 and by the fact that the multistep method (21) has the degree of exactness three, the following relations hold:

$$\mathcal{S}(x_k) = y(x_k) + O(h^3), \qquad \mathcal{S}''(x_k) = y''(x_k) + O(h^3). \tag{25}$$

The last relation results from Lemma 1 for $p = 3$.

LEMMA 4. *Let $y \in C^4[0, b]$ and assume x_k, $x_{k+1} = x_k + h$ to be in* [0, b]. *If P_3 is the unique polynomial of degree three satisfying the Hermite-Birkhoff interpolating condition*

$$\begin{aligned} P_3(x_k) &= y(x_k), & P_3''(x_k) &= y''(x_k), \\ P_3(x_{k+1}) &= y(x_{k+1}), & P_3''(x_{k+1}) &= y''(x_{k+1}), \end{aligned} \tag{26}$$

then there exists a constant K_3 such that

$$|P_3'''(x_k) - y'''(x_k)| < K_3 h.$$

Proof. If we write the cubic polynomial

$$P_3(x) = b_k + c_k(x - x_k) + d_k(x - x_k)^2 + e_k(x - x_k)^3$$

then conditions (26) give us

$$b_k = y(x_k), \qquad c_k = \frac{1}{h}[y(x_{k+1}) - y(x_k)] - \frac{h}{6}[y''(x_{k+1}) + 2y''(x_k)],$$

$$d_k = \tfrac{1}{2}y''(x_k), \qquad e_k = \frac{1}{6h}[y''(x_{k+1}) - y''(x_k)] = \tfrac{1}{6}y'''(\xi), \qquad x_k < \xi < x_{k+1}.$$

But $P_3'''(x) = P_3'''(x_k) = 6e_k = y'''(\xi)$. Consequently,

$$|P_3'''(x_k) - y'''(x_k)| = |y'''(\xi) - y'''(x_k)| = |\xi - x_k|\,|y^{(4)}(\eta)| < K_3 h, \qquad x_k < \eta < \xi$$

and the proof is completed.

THEOREM 4. *If $f \in C^3([0, b] \times R)$ and $\mathfrak{s}$ is the cubic spline function approximating the solution of problems* (6)–(7) *then there exists a constant K such that, for any $h < (6/A)^{1/2}$ and $x \in [0, b]$,*

$$|\mathfrak{s}(x) - y(x)| < Kh^3, \qquad |\mathfrak{s}'(x) - y'(x)| < Kh^2,$$

$$|\mathfrak{s}''(x) - y''(x)| < Kh^2, \qquad |\mathfrak{s}'''(x) - y'''(x)| < Kh,$$

provided $\mathfrak{s}'''(x_k)$ is given by (15) *with $m = 3$.*

Proof. Denote the cubic spline component over $[x_k, x_{k+1}]$ by

$$\mathfrak{s}(x) = b_k^{(1)} + c_k^{(1)}(x - x_k) + d_k^{(1)}(x - x_k)^2 + e_k^{(1)}(x - x_k)^3, \qquad x_k \leqq x \leqq x_{k+1} .$$

Solving a system similar to (26) for $\mathfrak{s}(x)$, we obtain

$$e_k^{(1)} = \frac{1}{6h}[\mathfrak{s}''(x_{k+1}) - \mathfrak{s}''(x_k)] = \frac{1}{6h}[y''(x_{k+1}) - y''(x_k)] + O(h^2)$$

$$= \tfrac{1}{6}P_3'''(x_k) + O(h^2)$$

since $\mathfrak{s}''(x_k) = y''(x_k) + O(h^3)$. Now let $x_k < x < x_{k+1}$. We have $\mathfrak{s}'''(x) = 6e_k^{(1)}$ and Lemma 4 implies

$$\mathfrak{s}'''(x) = P_3'''(x_k) + O(h) = y'''(x_k) + O(h) = y'''(x) + (x_k - x)y^{(4)}(\eta) + O(h).$$

Because $|x_k - x| < h$, we obtain

$$(27) \qquad \mathfrak{s}'''(x) = y'''(x) + O(h), \qquad x_k < x < x_{k+1}, k = 0, \cdots, n - 1.$$

Hence, it follows that condition (17) of Lemma 2 is satisfied for $m = 3$. Since the function $\mathfrak{s}'''$ is constant on (x_k, x_{k+1}), we may write

$$y(x_{k+1}) = y(x_k) + hy'(x_k) + \tfrac{1}{2}h^2y''(x_k) + \tfrac{1}{6}h^3y'''(\xi), \qquad x_k < \xi < x_{k+1} ,$$

$$\mathfrak{s}(x_{k+1}) = \mathfrak{s}(x_k) + h\mathfrak{s}'(x_k) + \tfrac{1}{2}h^2\mathfrak{s}''(x_k) + \tfrac{1}{6}h^3\mathfrak{s}'''(\xi).$$

Substracting we obtain

$$\begin{aligned}|\mathfrak{s}(x_{k+1}) - y(x_{k+1})| &= |\mathfrak{s}(x_k) - y(x_k) + h(\mathfrak{s}(x_k) - y'(x_k)) \\ &\quad + \tfrac{1}{2}h^2(\mathfrak{s}''(x_k) - y''(x_k)) + \tfrac{1}{6}h^3(\mathfrak{s}'''(\xi) - y'''(\xi))| \\ &= O(h^4).\end{aligned}$$

Relations (27), (25) imply that

$$(28) \qquad \mathfrak{s}'(x_k) - y'(x_k) = O(h^2).$$

From (25), (28) it follows that conditions (16) of Lemma 2 are fulfilled for $m = 3$, $p_0 = 3, p_1 = 2, p_2 = 3$. Note that $f \in C^3([0, b] \times R)$ implies $y \in C^4[0, b]$.

Applying Lemma 2 three times successively, first for $\mathfrak{s}$, and then for $\mathfrak{s}'$ and $\mathfrak{s}''$, the first three inequalities of the theorem follow. The last inequality follows from (20), and thus the theorem is proved.

4. Spline Function of Fourth Degree Approximating the Solution. If $m = 4$, Theorem 1 gives the following consistency relation for spline functions of degree four:

$$s_{k+1} - s_k - s_{k-1} + s_{k-2} = \frac{h^2}{12}[s''_{k+1} + 11s''_k + 11s''_{k-1} + s''_{k-2}], \quad 2 \leqq k \leqq n-1.$$

According to Theorem 3, the spline function of degree four approximating the solution furnishes values which, on the knots, coincide with the values of a discrete multistep method with the recurrence relation

$$\begin{aligned} y_{k+1} - y_k - y_{k-1} + y_{k-2} &= \frac{h^2}{12}[y''_{k+1} + 11y''_k + 11y''_{k-1} + y''_{k-2}] \\ &= \frac{h^2}{12}[f(x_{k+1}, y_{k+1}) + 11f(x_k, y_k) + 11f(x_{k-1}, y_{k-1}) + f(x_{k-2}, y_{k-2})], \end{aligned} \tag{29}$$

provided that the initial values are y_0, $y_1 = s(h)$, $y_2 = s(2h)$.

Multistep method (29) has degree of exactness five, if initial values have the same exactness (see [2, p. 295]).

LEMMA 5. *Let $m = 4$. Then, there is a constant K such that*

$$|s(h) - y(h)| < Kh^5 \quad \text{and} \quad |s(2h) - y(2h)| < Kh^5.$$

The proof parallels that of Lemma 3. The only difference consists in showing that $a_0 - y_0^{(4)} = O(h)$.

From the fact that the discrete method (29) has the degree of exactness five, and by Lemma 1 for $p = 5$, it follows that

$$s(x_k) - y(x_k) = O(h^5), \qquad s''(x_k) - y''(x_k) = O(h^5). \tag{30}$$

LEMMA 6. *Let $y \in C^5[0, b]$, and x_k, $x_{k+1} = x_k + h$ belong to $[0, b]$. If P_4 is the unique polynomial of degree four which satisfies the Hermite-Birkhoff interpolation conditions,*

$$\begin{aligned} &P_4(x_k) = y(x_k), \quad P_4(x_{k+1}) = y(x_{k+1}), \quad P_4''(x_k) = y''(x_k), \\ &P_4'''(x_k) = y'''(x_k), \quad P_4'''(x_{k+1}) = y'''(x_{k+1}), \end{aligned} \tag{31}$$

then there exists a constant K_4 such that

$$|P_4^{(4)}(x_k) - y^{(4)}(x_k)| < K_4 h.$$

The proof is similar to that of Lemma 4.

THEOREM 6. *If $f \in C^4([0, b] \times R)$ and s is the spline function of degree four approximating the solution y of (6)–(7), then there exists a constant K, such that, for any $h < (12/A)^{1/2}$, and $x \in [0, b]$,*

$$|s^{(j)}(x) - y^{(j)}(x)| < Kh^{5-j}, \qquad j = 0, \cdots, 4,$$

provided that $s^{(4)}(x_k)$ is calculated by (15) for $m = 4$.

Proof. On $[x_k, x_{k+1}]$, we write the spline function of degree four in the form

$$s(x) = b'_k + c'_k(x - x_k) + d'_k(x - x_k)^2 + e'_k(x - x_k)^3 + f'_k(x - x_k)^4, \quad x_k \leqq x \leqq x_{k+1}.$$

Since $s \in C^3[0, b]$, it follows by relations (30) that

$$s'''(x_k) - y'''(x_k) = O(h^4). \tag{32}$$

Solving (31) with $\mathcal{S}$ in place of P_4 we obtain for the coefficient f_k':

$$f_k' = \frac{1}{24h}[\mathcal{S}'''(x_{k+1}) - \mathcal{S}'''(x_k)]$$

$$= \frac{1}{24h}[y'''(x_{k+1}) - y'''(x_k)] + O(h^3)$$

$$= \frac{1}{24} P_4^{(4)}(x_k) + O(h^3),$$

where P_4 is the unique polynomial of degree four interpolating the data y_k, y_{k+1}, y_k'', y_k''', y_{k+1}''' taken from y.

Now let $x_k < x < x_{k+1}$. We have $\mathcal{S}^{(4)}(x) = 24f_k'$. By Lemma 6,

$$\mathcal{S}^{(4)}(x) = P_4^{(4)}(x_k) + O(h) = y^{(4)}(x_k) + O(h)$$

$$= y^{(4)}(x) + (x_k - x)y^{(5)}(\eta) + O(h), \qquad \eta \in (x_k, x).$$

Since $|x_k - x| < h$, it follows that

$$\mathcal{S}^{(4)}(x) = y^{(4)}(x) + O(h), \qquad x_k < x < x_{k+1}, k = 0, \cdots, n-1, \tag{33}$$

so that relation (17) of Lemma 2 is satisfied for $m = 4$.

Because $\mathcal{S}^{(4)}$ is constant on $[x_k, x_{k+1}]$ we can write

$$y(x_{k+1}) = y(x_k) + hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{3!}y'''(x_k) + \frac{h^4}{4!}y^{(4)}(\xi), \quad x_k < \xi < x_{k+1},$$

$$\mathcal{S}(x_{k+1}) = \mathcal{S}(x_k) + h\mathcal{S}'(x_k) + \frac{h^2}{2}\mathcal{S}''(x_k) + \frac{h^3}{3!}\mathcal{S}'''(x_k) + \frac{h^4}{4!}\mathcal{S}^{(4)}(\xi),$$

$$|\mathcal{S}(x_{k+1}) - y(x_{k+1})|$$

$$= \Big| \mathcal{S}(x_k) - y(x_k) + h(\mathcal{S}'(x_k) - y'(x_k)) + \frac{h^2}{2}(\mathcal{S}''(x_k) - y''(x_k))$$

$$+ \frac{h^3}{3!}(\mathcal{S}'''(x_k) - y'''(x_k)) + \frac{h^4}{4!}(\mathcal{S}^{(4)}(\xi) - y^{(4)}(\xi)) \Big| = O(h^5).$$

Relations (30), (32), (33) imply that

$$\mathcal{S}'(x_k) - y'(x_k) = O(h^4), \qquad k = 0, \cdots, n. \tag{34}$$

Relations (30), (32), (33), (34) show that the conditions of Lemma 2 are satisfied for $m = 4$, $p_0 = 5$, $p_1 = 4$, $p_2 = 5$, $p_3 = 4$. Obviously, from $f \in C^4([0, b] \times R)$, it follows that $y \in C^5[0, b]$.

Applying Lemma 2 for $\mathcal{S}$, then successively for $\mathcal{S}'$, $\mathcal{S}''$, $\mathcal{S}'''$, the theorem follows with the last relation coming from (20).

The method of approximating the solution of problems (6)–(7), by a spline function, given here for $m = 3, 4$, has the advantage over the discrete method that it gives a global approximation of the solution, is convergent and also permits the study of the behaviour of the derivatives of the approximate solution.

5. Instability of the Method for Splines of Degree $\geqq 5$.

THEOREM 7. *The approximate spline solution is divergent if $h \to 0$, for $m \geqq 5$.*

Let

$$\rho(z) = \sum_{k=0}^{m-1} c_k^{(m)} z^k$$

be the so-called characteristic polynomial attached to the discrete multistep method (13). By the theorem of Dahlquist [2, Theorem 6.1, p. 300], the discrete method (13) is stable only if the zeros of polynomial $\rho(z)$ do not exceed unity in modulus. Multiple zeros are not allowed to have greater multiplicity than 2. By (4) and taking into account the properties of the B-spline (see [3, p. 19]), it follows at once that

$$\begin{aligned}\rho(z) &= \sum_{k=0}^{m-1} (m-1)!\, Q_m(k+1) z^k \\ &= (m-1)(z-1)^2 \Big\{ z^{m-3} + (2^{m-2} - m + 1) z^{m-4} \\ &\qquad + \left[3^{m-2} - (m-1)2^{m-2} + \frac{(m-1)(m-2)}{2} \right] z^{m-5} + \cdots + 1 \Big\} \\ &= (m-1)(z-1)^2 \rho_1(z).\end{aligned}$$

If we denote the roots of ρ_1 by $z_3, z_4, \cdots, z_{m-1}$, then

$$\sum_{k=3}^{m-1} z_k = m - 1 - 2^{m-2}.$$

Hence, it follows that

$$\sum_{k=3}^{m-1} |z_k| \geqq \left| \sum_{k=3}^{m-1} z_k \right| = 2^{m-2} - m + 1 > m - 2 \quad \text{if } m \geqq 5.$$

If we set $Z_M = \max_k |z_k|$, then

$$(m-3) Z_M > m - 2 \quad \text{or} \quad Z_M > (m-2)/(m-3) > 1 \quad \text{if } m \geqq 5.$$

Thus, the multistep method and, hence, the corresponding spline solution are divergent.

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