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An Elliptic Integral Identity

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Abstract. The identity

$$K(\tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2 + y^4)\right] dx\, dy,$$

where $K(\tau)$ is the complete elliptic integral of the first kind, is used to prove that $K(\sqrt{2} - 1) = \pi^{3/2}(2 + \sqrt{2})^{1/2}/4\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})$.

The identity

$$\text{(1)}\quad K(\tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left[-\tfrac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2 + y^4)\right] dx\, dy, \quad |\tau| < 1,$$

is easily proved by using the standard transformation $x = r\cos v$, $y = r \sin v$. Before proceeding, we will list a few identities for the benefit of the reader.

$$\text{(2)}\qquad \int_0^{\infty} e^{-pt} t^{-1/2} e^{-t^2/4\alpha}\, dt = \alpha^{1/2} p^{1/2} e^{\alpha p^2/2} K_{1/4}(\tfrac{1}{2}\alpha p^2), \quad \operatorname{Re}\alpha > 0 \quad [1, \text{p. } 146],$$

$$\text{(3)}\qquad \int_0^{\infty} e^{-pt} t^{\mu - 1/2} K_{\nu + 1/2}(\alpha t)\, dt = 2^{-1/2}\alpha^{-1/2}\pi^{1/2}\Gamma(\mu - \nu)\Gamma(\mu + \nu + 1) s^{-\mu} P_\nu^{-\mu}(p/\alpha),$$

$$s = (p^2 - \alpha^2)^{1/2}, \quad \operatorname{Re}(\mu + \nu) > -1, \quad \operatorname{Re}(\mu - \nu) > 0,$$

$$\operatorname{Re} p > -\operatorname{Re}\alpha, \qquad [1, \text{p. } 198],$$

$$\text{(4)}\qquad P_p^q(0) = \frac{2^q \pi^{1/2}}{\Gamma(\frac{1}{2}(p - q) + 1)\Gamma(\frac{1}{2}(-p - q + 1))} \qquad [2, \text{p. } 354].$$

We will now calculate the integral (1) in a different way.

Integration with respect to x yields

$$\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2)\right] dx = \int_0^{\infty} t^{-1/2} e^{-t^2/2} e^{(2\tau^2 - 1)y^2 t}\, dt$$

$$= \frac{(1 - 2\tau^2)^{1/2}}{2^{1/2}} |y| \exp\left[\frac{(2\tau^2 - 1)^2 y^4}{4}\right] K_{1/4}\left(\frac{(2\tau^2 - 1)^2 y^4}{4}\right), \qquad 1 - 2\tau^2 > 0.$$

Integration with respect to y now gives

$$K(\tau) = \frac{(1 - 2\tau^2)^{1/2}}{2\pi^{1/2}} \int_{-\infty}^{+\infty} |y|\, e^{-y^4/2} \exp\left[\frac{(2\tau^2 - 1)^2 y^4}{4}\right] K_{1/4}\left(\frac{(2\tau^2 - 1)^2 y^4}{4}\right) dy.$$

A few manipulations finally give

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$$K(\tau) = \frac{(1-2\tau^2)^{1/2}}{2\pi^{1/2}} \int_0^\infty t^{-1/2} e^{-t(1+4\tau^2-4\tau^4)} K_{1/4}((1-2\tau^2)^2 t)\, dt. \tag{5}$$

Using Eq. (3) we can write

$$K(\tau) = \frac{\pi}{2(1-2\tau^2)^{1/2}} P_{-1/4}\left(\frac{1+4\tau^2-4\tau^4}{(1-2\tau^2)^2}\right). \tag{6}$$

To get (6) in a more convenient form, we use the well-known formula

$$K(i\tau) = \frac{1}{(1+\tau^2)^{1/2}} K\left(\frac{\tau}{(1+\tau^2)^{1/2}}\right).$$

Then

$$K\left(\frac{\tau}{(1+\tau^2)^{1/2}}\right) = \frac{\pi(1+\tau^2)^{1/2}}{2(1+2\tau^2)^{1/2}} P_{-1/4}\left(\frac{1-4\tau^2-4\tau^4}{(1+2\tau^2)^2}\right). \tag{7}$$

Now put $1 - 4\tau^2 - 4\tau^4 = 0$ which yields $\tau^2 = -\frac{1}{2} \overset{+}{(-)} \frac{1}{2}\sqrt{2}$. A little calculation and Eq. (4) give

$$K(\sqrt{2}-1) = \frac{\pi^{3/2}(2+\sqrt{2})^{1/2}}{4\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}. \qquad \text{Q.E.D.} \tag{8}$$

The identity (8) has been proved previously; see for instance [3, pp. 535–536]. The present derivation seems, however, to give a better insight into the problem.

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