MATHEMATICS OF COMPUTATION, VOLUME 27, NUMBER 124, OCTOBER 1973

A Note on the Computation of Integrals Involving Products of Trigonometric and Bessel Functions

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Abstract. A method useful in the numerical computation of integrals involving Bessel functions is extended to the case where the integrand contains products of trigonometric and Bessel functions.

In a previous paper [1], one of us presented a method suitable for the numerical computation of integrals of the form

$$I(\rho) = \int_0^\infty f(x) J_0(\rho x)\, dx. \tag{1}$$

It is the purpose of this note to outline some extensions of the technique in [1] to more complicated integrals such as

$$A(\rho, \tau) = \int_0^\infty f(x) J_0(\rho x) \cos(\tau x)\, dx \tag{2}$$

and

$$B(\rho, \tau) = \int_0^\infty f(x) J_0(\rho x) J_0(\tau x)\, dx. \tag{3}$$

The method is based on the observation that the application of an Abel integral transform converts Bessel functions into trigonometric functions and thus integrals involving Bessel functions into more easily treated Fourier integrals. In [1], it was shown that

$$I(\rho) = \frac{2}{\pi} \int_0^\rho (\rho^2 - s^2)^{-1/2} C(s)\, ds, \tag{4}$$

where

$$C(s) \equiv \int_0^\infty f(x) \cos(xs)\, dx. \tag{5}$$

Applying the same technique to (2), and assuming $f(x)$ is such that all formal manipulations below are valid, we get

Received January 26, 1973.

AMS (MOS) subject classifications (1970). Primary 65D30; Secondary 35A40, 42A68, 44A20.

Key words and phrases. Bessel functions, Fourier-Bessel coefficients, Hankel transforms, numerical integration.

$$
\begin{aligned}
D(r, \tau) &\equiv \int_0^r \rho(r^2 - \rho^2)^{-1/2} A(\rho, \tau)\, d\rho \\
&= \int_0^r \rho(r^2 - \rho^2)^{-1/2} \int_0^\infty f(x) J_0(\rho x) \cos(\tau x)\, dx\, d\rho \\
&= \int_0^\infty f(x) \cos \tau x\, dx \int_0^r \rho(r^2 - \rho^2)^{-1/2} J_0(\rho x)\, d\rho \\
&= \int_0^\infty \frac{1}{x} f(x) \cos \tau x \sin xr\, dx.
\end{aligned}
$$

Then

$$
\frac{\partial}{\partial r} D(r, \tau) = \int_0^\infty f(x) \cos \tau x \cos xr\, dx = \tfrac{1}{2}\{C(\tau - r) + C(\tau + r)\}.
$$

Using the Abel inversion formula,

$$
A(\rho, \tau) = \frac{1}{\pi} \int_0^\rho (\rho^2 - r^2)^{-1/2}\{C(\tau - r) + C(\tau + r)\}\, dr. \tag{6}
$$

Applying the same technique to (3), we can show that

$$
B(\rho, \tau) = \frac{2}{\pi} \int_0^\tau (\tau^2 - s^2)^{-1/2} A(\rho, s)\, ds. \tag{7}
$$

The computation of $A(\rho, \tau)$ and $B(\rho, \tau)$ thus requires the evaluation of some Fourier cosine integrals and an integration; this will generally be done numerically. The numerical computation of the Fourier integrals is a classical problem and a number of algorithms are available. Product integration techniques can be used for treating the integrals in (6) and (7); for details see [1].

The above approach is particularly efficient if $A(\rho, \tau)$ and $B(\rho, \tau)$ are needed for a range of ρ and τ. For instance, if we want to compute $A(\rho, \tau)$ for $0 \leq \rho \leq R$, $0 \leq \tau \leq T$, then we need $C(x)$ only for $-R \leq x \leq R + T$. To compute $B(\rho, \tau)$ for the same range, we need $A(\rho, s)$ for $0 \leq \rho \leq R$, $0 \leq s \leq T$, i.e., $C(x)$ for $-R \leq x \leq R + T$. This allows us to compute $A(\rho, \tau)$ and $B(\rho, \tau)$ over a rectangle with just a few numerical values of the Fourier integral.

Generalizations of this procedure to integrals with Bessel functions of higher order can be made along the lines indicated in [1]. Integrals involving products of several trigonometric and Bessel functions can be treated by repeated application of the Abel transform.

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1. P. Linz, "A method for computing Bessel function integrals," *Math. Comp.*, v. 26, 1972, pp. 509–513.