On the Largest Prime Divisor of an Odd Perfect Number

By Peter Hagis, Jr. and Wayne L. McDaniel

Abstract. It is shown here that if n is odd and perfect, then n has a prime divisor which exceeds 11200.

1. Introduction. If $\sigma(n)$ denotes the sum of the positive divisors of the natural number n, then n is said to be perfect if

(1) $$\sigma(n) = 2n.$$

Twenty-four even perfect numbers are known at present, but whether or not an odd perfect number exists is an open question. However, many necessary conditions for the existence of an odd perfect number are known. The oldest of these is due to Euler who proved that if n is odd and perfect, then

(2) $$n = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_t^{\alpha_t}$$

where the p_i are distinct primes, $p_0 \equiv \alpha_0 \equiv 1 \pmod 4$, and α_i is even for $i > 0$. In 1944 Kanold [2] showed that at least one of the p_i is greater than or equal to 61. Our purpose here is to show that Kanold's result can be significantly improved. Thus, we shall prove the following result.

THEOREM. *If n is an odd perfect number, then n is divisible by a prime which exceeds* 11200.

Our proof is computational in nature in as much as it is based on a search for the prime factors less than 11200 of $\sigma(p^\alpha)$ for a large number of values of p and α. This search was carried out using the CDC 6400 at the Temple University Computing Center and utilized the fact that $\sigma(p^\alpha) = \prod F_d(p)$ where F_d is the dth cyclotomic polynomial and d runs over the divisors of $\alpha + 1$ which exceed 1. The theorems concerning the prime divisors of cyclotomic polynomials which may be found in Section 48 of [3] facilitate the search.

2. Some Preliminaries. According to a theorem of Kanold [1], if P is the greatest of the p_i in (2) and β is the maximum of the α_i, then $P > 2(\beta + 1)$. It follows that for our purpose we may assume that each α_i is less than 5599. Thus, in the remainder of this paper, n will always denote an odd perfect number whose canonical factorization is given by (2), with every $\alpha_i < 5599$.

If $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$, we shall write $p^\alpha \parallel n$. In this case, we note from (1) and the fact that the σ-function is multiplicative that every odd prime which divides $\sigma(p^\alpha)$ also divides n.

3. Phase I. Suppose that p is a prime and α is a natural number. We shall say that p^α is (5599; 11200)-acceptable or simply *acceptable* if $\alpha < 5599$ and every prime

which divides $\sigma(p^\alpha)$ is less than 11200. (We consider only even values of α unless $p = 4k + 1$ in which case we also consider those α of the form $4k + 1$.) For example, since $\sigma(31^\alpha)$ is divisible by a prime greater than 11200 for $\alpha = 4, 6, 8, \cdots, 5598$ the only acceptable power of 31 is 31^2. Clearly, our theorem is true if at least one of the $p_i^{\alpha_i}$ in (2) is *unacceptable*. Assuming that each prime power in (2) *is* acceptable we shall now show that n is not divisible by certain "small" primes.

(A) $31 \nmid n$. For the only acceptable power of 31 is 31^2. Since $\sigma(31^2) = 3 \cdot 331$, we see that if $31 \mid n$ then also $331 \mid n$. Only 331^2 is acceptable. But $5233 \mid \sigma(331^2)$, and only 5233 is acceptable. $2617 \mid \sigma(5233)$ and only 2617 is acceptable. We conclude that $5233 \parallel n$ and $2617 \parallel n$ which contradicts (2).

(B) $67 \nmid n$. For only 67^2 is acceptable and $\sigma(67^2) = 3 \cdot 7^2 \cdot 31$. Therefore, if $67 \mid n$ then $31 \mid n$. This contradicts (A).

(C) $19 \nmid n$. The only acceptable powers of 19 are 19^2 and 19^4. But $911 \mid \sigma(19^4)$, and no power of 911 is acceptable. $\sigma(19^2) = 3 \cdot 127$ and only 127^2 is acceptable. $5419 \mid \sigma(127^2)$ and only 5419^2 is acceptable. Since $31 \mid \sigma(5419^2)$, $19^2 \parallel n$ is impossible.

(D) $11 \nmid n$. Only 11^2 and 11^4 are acceptable. If $11^2 \parallel n$ then, since $\sigma(11^2) = 7 \cdot 19$, $19 \mid n$ which contradicts (C). If $11^4 \parallel n$ then $5 \cdot 3221 \mid n$. Only 3221 is acceptable and $\sigma(3221) = 2 \cdot 3^2 \cdot 179$. Only 179^2 is acceptable and $7 \mid \sigma(179^2)$. Therefore, $3 \cdot 5 \cdot 7 \mid n$ which is impossible. (See (g) on p. 109 of [1].)

(E) $37 \nmid n$. Only 37, 37^2, 37^4, 37^5 are acceptable. But $19 \mid \sigma(37)$; $67 \mid \sigma(37^2)$; $11 \mid \sigma(37^4)$; $19 \mid \sigma(37^5)$.

(F) If $29 \mid n$, then $29 \parallel n$. For the only acceptable powers of 29 are 29, 29^2, 29^5 and 67 divides both $\sigma(29^2)$ and $\sigma(29^5)$.

(G) $7 \nmid n$. The acceptable powers of 7 are 7^2, 7^4, 7^6, 7^8. Both $\sigma(7^2)$ and $\sigma(7^8)$ are divisible by 19. Therefore, $7^2 \nmid n$ and $7^8 \nmid n$. If $7^4 \parallel n$ then $2801 \mid n$. Only 2801 and 2801^2 are acceptable, and $37 \mid \sigma(2801^2)$. $467 \mid \sigma(2801)$ and no power of 467 is acceptable. We conclude that $7^4 \nmid n$. $\sigma(7^6) = 29 \cdot 4733$, and only 4733 is acceptable. If $7^6 \parallel n$, it then follows from (F) that $29 \parallel n$ and $4733 \parallel n$. This contradicts (2).

(H) $23 \nmid n$, $47 \nmid n$, $83 \nmid n$. For each of the primes 23, 47, 83, only the second power is acceptable. But $7 \mid \sigma(23^2)$; $37 \mid \sigma(47^2)$; $19 \mid \sigma(83^2)$.

(I) $11197 \nmid n$. For only 11197 is acceptable and $11 \mid \sigma(11197)$.

We collect these results as a lemma.

LEMMA. *If every prime power in the canonical factorization of the odd perfect number n is acceptable, then n is not divisible by any prime in the set A where $A = \{7, 11, 19, 23, 31, 37, 47, 67, 83\}$.*

4. **Phase II.**   Let q be the smallest prime which divides n, and let $B(q)$ denote the set of odd primes which are less than q. If $p \notin A$ we shall say that p^α is (5599; 11200; A; q)-feasible or feasible with respect to q or simply *feasible* if p^α is acceptable and if also no prime divisor of $\sigma(p^\alpha)$ belongs to the set $A \cup B(q)$. Our theorem is obviously true if p^α is not feasible and $p^\alpha \parallel n$.

If $q \geq 101$ then, according to the table in [4], $p \mid n$ where $p \geq 11197$. Referring to (I) and our Lemma, we see then that for our purpose we can assume that q is in the set $S = \{3, 5, 13, 17, 29, 41, 43, 53, 59, 61, 71, 73, 79, 89, 97\}$. A computer search revealed that, unless $q \in T$ where $T = \{3, 17, 41, 59, 71, 79, 89\}$, then q has no feasible powers with respect to q. Therefore, our theorem is certainly true unless q, the smallest prime dividing n, belongs to T and every prime power in the canonical factorization of

n is feasible with respect to q. We shall complete our proof by showing that for each prime in T the assertion that the prime in question is q is incompatible with the requirement of feasibility for each of the $p_i^{\alpha_i}$ in (2).

(a) $q \neq 3$. The only feasible powers of 3 are $3^2, 3^6, 3^8, 3^{26}$. If $3^2 \parallel n$ then $13 \mid n$. The only feasible power of 13 is 13^2, and $\sigma(13^2) = 3 \cdot 61$. Only 61^2 is feasible, and $\sigma(61^2) = 3 \cdot 13 \cdot 97$. Only 97^2 is feasible, and $\sigma(97^2) = 3 \cdot 3169$. Therefore, $3^3 \mid n$ which contradicts $3^2 \parallel n$. $\sigma(3^6) = 1093$, and only 1093 is feasible. $547 \mid \sigma(1093)$, and only 547^2 is feasible. Since $163 \mid \sigma(547^2)$ and no power of 163 is feasible, we see that $3^6 \nmid n$. 757 divides both $\sigma(3^8)$ and $\sigma(3^{26})$. Only 757 is feasible and $379 \mid \sigma(757)$. Only 379^2 is feasible and $787 \mid \sigma(379^2)$. No power of 787 is feasible. Therefore, $3^8 \nmid n$ and $3^{26} \nmid n$.

(b) $q \neq 17$. 17^2 is the only feasible power of 17. $\sigma(17^2) = 307$, and only 307^4 is feasible. But $1051 \mid \sigma(307^4)$, and 1051 has no feasible powers. We conclude that $17^2 \nmid n$.

(c) $q \neq 41$. Only 41^2 is feasible, and $\sigma(41^2) = 1723$. 1723 has no feasible powers.

(d) $q \neq 59$. 59^2 is the only feasible power of 59. But $\sigma(59^2) = 3541$, and 3541 has no feasible powers.

(e) $q \neq 71$. The only feasible power of 71 is 71^2. $\sigma(71^2) = 5113$, and only 5113 is feasible. $2557 \mid \sigma(5113)$ and only 2557 is feasible. Therefore, $5113 \parallel n$ and $2557 \parallel n$ which contradicts (2).

(f) $q \neq 79$. Only 79^6 is feasible. But $1289 \mid \sigma(79^6)$, and 1289 has no feasible powers.

(g) $q \neq 89$. 89^2 and 89^4 are both feasible. But $\sigma(89^2) = 8011$ and $131 \mid \sigma(89^4)$; and neither 8011 nor 131 have any feasible powers.

5. Concluding Remark.

It seems very likely that the attack of this paper can be used to obtain a much greater lower bound than 11213 for the maximum prime divisor of an odd perfect number. However, both the effort and the computer time required appear to increase at an exponential rate.

Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122

Department of Mathematics
University of Missouri-St. Louis
St. Louis, Missouri 63121

1. H.-J. KANOLD, "Untersuchungen über ungerade vollkommene Zahlen," *J. Reine Angew. Math.*, v. 183, 1941, pp. 98–109. MR **3,** 268.

2. H.-J. KANOLD, "Folgerungen aus dem Vorkommen einer Gauss'schen Primzahl in der Primfaktorenzerlegung einer ungeraden vollkommenen Zahl," *J. Reine Angew. Math.*, v. 186, 1944, pp. 25–29. MR **6,** 255.

3. T. NAGELL, *Introduction to Number Theory*, Wiley, New York; Almqvist & Wiksell, Stockholm, 1951. MR **13,** 207.

4. K. NORTON, "Remarks on the number of factors of an odd perfect number," *Acta Arith.*, v. 6, 1960, pp. 365–374. MR **26** #4950.