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Numerical Solution of Lundquist Equations of Magnetohydrodynamics*

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Abstract. A method of bicharacteristics [3] is used to derive a numerical method for solving multidimensional nonlinear Lundquist equations of magnetohydrodynamics. Actual numerical computations are carried out to solve two specific problems of magnetohydrodynamics—the magnetohydrodynamic initial-pressure problem and a problem of cylindrical waves in a transverse magnetic field due to a thin current-carrying wire perpendicular to the plane of the fluid.

1. Introduction. In this paper, we derive a method for the numerical solution of Lundquist equations which describe the flow of an electrically conducting fluid in the presence of an electromagnetic field. The method is derived from a method of bicharacteristics. In deriving the method, we have used the most general three-dimensional situation without linearization. The method is then used to solve two specific problems in magnetohydrodynamics—the magnetohydrodynamic initial-pressure problem and the problem of cylindrical waves in a transverse magnetic field due to a thin current-carrying wire perpendicular to the plane of the fluid. The exact solution of the initial-pressure problem in the presence of a uniform magnetic field was obtained by Friedlander [1] from the linearized version of the Lundquist equations. We compute solutions for both nonlinear and linearized equations.

2. The Lundquist Equations [2]. The equations of motion of magnetohydrodynamics consist of two groups. First, there are the equations of motion of an inviscid, electrically conducting fluid,

$$\rho\{\partial V/\partial t + (V\cdot\operatorname{grad})V\} + \operatorname{grad} p - \mu j \times H = 0,$$

$$\partial \rho/\partial t + (V\cdot\operatorname{grad})\rho + \rho \operatorname{div} V = 0,$$

where p is the pressure, ρ the density, V the velocity, j the current, H the magnetic field and μ the permeability. Secondly, one has Maxwell's equations

$$\operatorname{curl} H = 4\pi j, \quad \operatorname{curl} E = -\mu\, \partial H/\partial t,$$

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where E is the field strength and the displacement current has been neglected. Furthermore, assuming that the fluid is a perfect conductor,

$$E + \mu V \times H = 0.$$

Eliminating E, one obtains

$$\operatorname{curl} H = 4\pi j, \quad \partial H/\partial t = \operatorname{curl}(V \times H).$$

These equations must, as usual, be supplemented by div $H = 0$.

Finally, there is the energy equation. Since we have already neglected viscosity and taken the electrical conductivity to be infinite, we must, for the sake of consistency, also ignore heat conduction. Then the entropy of any fluid particle remains constant and this implies that

$$\partial p/\partial t + (V \cdot \mathbf{grad})p = C^2\{\partial \rho/\partial t + (V \cdot \mathbf{grad})\rho\},$$

where $C^2 = (\partial p/\partial \rho)^{1/2}$, evaluated for constant specific entropy, is the local velocity of sound. In this case, the pressure-density relation is

$$p = A\rho^\gamma, \qquad A = \text{constant}.$$

Thus the equations for p, V and H are

$$U_t + A_1 U_x + A_2 U_y + A_3 U_z = 0 \tag{2.1}$$

where

$$U = [p, V_x, V_y, V_z, H_x, H_y, H_z]^T, \tag{2.2}$$

$$A_1 = \begin{bmatrix} V_x & C^2\rho & 0 & 0 & 0 & 0 & 0 \\ 1/\rho & V_x & 0 & 0 & 0 & KH_y & KH_z \\ 0 & 0 & V_x & 0 & 0 & -KH_x & 0 \\ 0 & 0 & 0 & V_x & 0 & 0 & -KH_x \\ 0 & 0 & 0 & 0 & V_x & 0 & 0 \\ 0 & H_y & -H_x & 0 & 0 & V_x & 0 \\ 0 & H_z & 0 & -H_x & 0 & 0 & V_x \end{bmatrix}, \tag{2.3}$$

$$A_2 = \begin{bmatrix} V_y & 0 & C^2\rho & 0 & 0 & 0 & 0 \\ 0 & V_y & 0 & 0 & -KH_y & 0 & 0 \\ 1/\rho & 0 & V_y & 0 & KH_x & 0 & KH_z \\ 0 & 0 & 0 & V_y & 0 & 0 & -KH_y \\ 0 & -H_y & H_x & 0 & V_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_y & 0 \\ 0 & 0 & H_z & -H_y & 0 & 0 & V_y \end{bmatrix}, \tag{2.4}$$

$$(2.5) \qquad A_3 = \begin{bmatrix} V_z & 0 & 0 & C^2\rho & 0 & 0 & 0 \\ 0 & V_z & 0 & 0 & -KH_z & 0 & 0 \\ 0 & 0 & V_z & 0 & 0 & -KH_z & 0 \\ 1/\rho & 0 & 0 & V_z & KH_x & KH_y & 0 \\ 0 & -H_z & 0 & H_x & V_z & 0 & 0 \\ 0 & 0 & -H_z & H_y & 0 & V_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_z \end{bmatrix},$$

where $K = \mu/4\pi\rho$.

3. The Method. The method used for the Lundquist equations is the method of bicharacteristics developed by Johnston and Pal [3] for a hyperbolic system of partial differential equations. We are given a hyperbolic system

$$(3.1) \qquad U_t + \sum_{j=1}^{m} A_j U_{x_j} = 0$$

where $U = [u_1, u_2, \cdots, u_n]^T$ and A_j's are $N \times N$ matrices. The bicharacteristics [2] of the system are given by

$$(3.2) \qquad dx_i/dt = \partial h_j/\partial \lambda_i, \qquad 1 \leqq j \leqq N, 1 \leqq i \leqq m,$$

$$(3.3) \qquad d\lambda_i/dt = -\partial h_j/\partial x_i, \qquad 1 \leqq j \leqq N, 1 \leqq i \leqq m,$$

where $-h_j$, $1 \leqq j \leqq N$, are the N real zeros of the characteristic polynomial

$$(3.4) \qquad H = \det\left(\lambda I + \sum_{j=1}^{m} A_j \lambda_j\right)$$

for a given set $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ of real numbers. Selecting a bicharacteristic and integrating along it, the system of given differential equations can be written as the following equivalent system:

$$(3.5) \qquad U(t + \Delta t, \xi) = U(t, \alpha) + \sum_{j=1}^{m} \int_t^{t+\Delta t} D_j U_{x_j} \, dt$$

where $D_j = -A_j + (\partial h/\partial \lambda_j)I$, h being an eigenvalue of the matrix $\sum A_j \lambda_j$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is given by

$$(3.6) \qquad \alpha_i = \xi_i - \int_t^{t+\Delta t} \frac{\partial h}{\partial \lambda_i} dt, \qquad 1 \leqq i \leqq m.$$

Approximating the integrals using $\int_t^{t+\Delta t} f(x, s)\, ds \simeq f(x, t)\Delta t$ and the derivatives U_{x_j} by

$$(3.7) \qquad U_{x_j}(t, \xi) \simeq (1/2\Delta x_j)[E_{+j} - E_{-j}]\, U(t, \xi),$$

we get an explicit method

$$U^{n+1}(\xi) = \left(1 - \sum_{j=1}^{m} \left|\frac{\partial h}{\partial \lambda_j}\right| r_j\right)_Q U^n(\xi) + \frac{1}{2}\sum_{j=1}^{m} r_j\left(-A_j + \left|\frac{\partial h}{\partial \lambda_j}\right| I\right)_Q E_{+j} U^n(\xi) \tag{3.8}$$
$$+ \frac{1}{2}\sum_{j=1}^{m} r_j\left(A_j + \left|\frac{\partial h}{\partial \lambda_j}\right| I\right)_Q E_{-j} U^n(\xi).$$

Using the approximation

$$\int_t^{t+\Delta t} f(x, s)\, ds \simeq \tfrac{1}{2}[f(x, t) + f(x, t + \Delta t)]\, \Delta t$$

for the integrals and the same approximations as in (3.7) for the derivatives U_{x_j}, we have an implicit method

$$U^{n+1}(\xi) = \left(1 - \sum_{j=1}^{m} \left|\frac{\partial h}{\partial \lambda_j}\right| r_j\right)_Q U^n(\xi) + \sum_{j=1}^{m} \left|\frac{\partial h}{\partial \lambda_j}\right|_Q r_j E_{\pm j} U^n(\xi)$$
$$+ \frac{1}{4}\sum_{j=1}^{m} \left(-A_j + \frac{\partial h}{\partial \lambda_j} I\right)_Q r_j [E_{+j} - E_{-j}] U^n(\xi) \tag{3.9}$$
$$+ \frac{1}{4}\sum_{j=1}^{m} \left(-A_j + \frac{\partial h}{\partial \lambda_j} I\right)_Q r_j [E_{+j} - E_{-j}] U^{n+1}(\xi).$$

Here $r_j = \Delta t/\Delta x_j$, Q is the point $(n\Delta t, \alpha)$ and

$$E_{\pm j} f(t, x) = f(t, x_1, x_2, \cdots, x_2, \cdots, x_j \pm \Delta x_j, \cdots, x_m)$$

and $E_{\pm j}$ is understood to mean E_{+j} when $(\partial h/\partial \lambda_j)_Q \leqq 0$ and E_{-j} when $(\partial h/\partial \lambda_j)_Q > 0$.

It is natural to ask if the choice of different bicharacteristics makes any significant difference in the solution obtained. The question is answered in [3] where it is shown that the discrepancy in the solutions obtained by using two different bicharacteristics is at most of the same order as the truncation error which is $O(\Delta t^2)$.

In [3] it is also shown that the methods are stable provided that λ_j's are chosen such that $|\partial h/\partial \lambda_j| \leqq \max_i \|A_i\|_2 \leqq R$ and $r_j = \Delta t/\Delta x_j < \Omega/mR^2$, $1 \leqq j \leqq m$, where $\Omega = \min_j |\partial h/\partial \lambda_j|$. The eigenvalues of $A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3$ for the Lundquist equations are

(1) $V_x\lambda_1 + V_y\lambda_2 + V_z\lambda_3$.

(2) & (3) $V_x\lambda_1 + V_y\lambda_2 + V_z\lambda_3 \pm c_f \cdot \lambda$,

$$c_f^2 = \tfrac{1}{2}[(c^2 + b^2) + \{(c^2 + b^2)^2 - 4c^2 b_n^2\}^{1/2}].$$

(4) & (5) $V_x\lambda_1 + V_y\lambda_2 + V_z\lambda_3 \pm c_s \cdot \lambda$,

$$c_s^2 = \tfrac{1}{2}[(c^2 + b^2) - \{(c^2 + b^2)^2 - 4c^2 b_n^2\}^{1/2}].$$

(6) & (7) $V_x\lambda_1 + V_y\lambda_2 + V_z\lambda_3 \pm b_n\lambda$, where

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$

$$b^2 = (\mu/4\pi\rho)(H_x^2 + H_y^2 + H_z^2), \qquad b_n = (\mu/4\pi\rho)^{1/2}(H_x\lambda_1 + H_y\lambda_2 + H_z\lambda_3)/\lambda,$$

$$H_n = (H_x\lambda_1 + H_y\lambda_2 + H_z\lambda_3)/\lambda, \quad H^2 = H_x^2 + H_y^2 + H_z^2.$$

Here the bicharacteristics corresponding to the eigenvalue $h = V_x\lambda_1 + V_y\lambda_2 + V_z\lambda_3 + c_f \cdot \lambda$ are determined by

$$\frac{dx}{dt} = \frac{\partial h}{\partial \lambda_1} = V_x + c_f\lambda_1/\lambda - \frac{b^2 s(H_n/H^2)}{c_f C}(H_x - H_n\lambda_1/\lambda) ,$$

$$\frac{dy}{dt} = \frac{\partial h}{\partial \lambda_2} = V_y + c_f\lambda_2/\lambda - \frac{b^2 s(H_n/H^2)}{c_f C}(H_y - H_n\lambda_2/\lambda) ,$$

$$\frac{dz}{dt} = \frac{\partial h}{\partial \lambda_3} = V_z + c_f\lambda_3/\lambda - \frac{b^2 s(H_n/H^2)}{c_f C}(H_z - H_n\lambda_3/\lambda) ,$$

where

$$s = c^2/b^2, \quad C = [(1 + s)^2 - 4sb_n^2/b^2]^{1/2}.$$

Similarly, bicharacteristics corresponding to the other eigenvalues can be obtained. Numerical solution can now be obtained by using these values of $\partial h/\partial \lambda_1$, $\partial h/\partial \lambda_2$, $\partial h/\partial \lambda_3$ in either (3.8) or (3.9).

Here, it should be noted that in the derivation of the method no assumption was made about the strictly hyperbolic nature of the equations. Thus even if, in a given case, certain bicharacteristic rays are taken more than once, the method should be successful as is proved by the computational results obtained in the next section in the two degenerate cases where (1) $c_s{}^2 = 0$ and the wave-speed locus has a double point and (2) $c_f = b, -b, c_s = b, -b$.

4. The Numerical Solutions.

(a) *The Magnetohydrodynamic Initial-Pressure Problem.* This problem, as mentioned before, was considered by Friedlander [1]. It is a problem in three space dimensions and concerns the propagation of small disturbances in a compressible fluid which is also a conductor of electricity in the presence of a magnetic field. Energy dissipation by viscosity, heat conduction and the Joule heat is neglected as is the displacement current. The equations governing the phenomenon are the Lundquist equations (2.1). We now consider departures from an equilibrium in which the medium is uniform. Let p_0, ρ_0 and H_0 be the constant equilibrium values of the pressure, density and the magnetic field respectively and put $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, $H = H_0 + H'$, where p', ρ' and H' are departures from the respective equilibrium values. Then the equations for p', V, H' are

$$U_t + A_1 U_x + A_2 U_y + A_3 U_z = 0 \tag{4.1}$$

where

$$U = [p', V_x, V_y, V_z, H_x', H_y', H_z']^T, \tag{4.2}$$

and A_1, A_2, A_3 are as defined in (2.3), (2.4) and (2.5). Assuming small departures, and by choosing the direction of H_0 as the x-axis, we can obtain an approximation to the system by the following linear system

$$\frac{\partial p'}{\partial t} + c^2\rho_0\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = 0, \tag{4.3}$$

$$\frac{\partial V_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0, \tag{4.4}$$

$$\frac{\partial V_y}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} + K_0 H_0\left[\frac{\partial H_x'}{\partial y} - \frac{\partial H_y'}{\partial x}\right] = 0, \tag{4.5}$$

$$\frac{\partial V_z}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial z} + K_0 H_0\left[\frac{\partial H_x'}{\partial z} - \frac{\partial H_z'}{\partial x}\right] = 0, \tag{4.6}$$

$$\frac{\partial H_x'}{\partial t} + H_0\left[\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] = 0, \tag{4.7}$$

$$\frac{\partial H_y'}{\partial t} - H_0\frac{\partial V_y}{\partial x} = 0, \tag{4.8}$$

$$\frac{\partial H_z'}{\partial t} - H_0\frac{\partial V_z}{\partial x} = 0, \tag{4.9}$$

where $p' = c^2\rho'$, c being now the velocity of sound calculated with the equilibrium p_0 and ρ_0 and $K_0 = \mu/4\pi\rho_0$. These equations, of course, follow directly from the

TABLE 1

Solution of Initial-Pressure Problem at $x = y = z = .4$. 1 and 11 denote the exact and numerical solutions, respectively, of the linearized problem while 111 denotes the numerical solution of the nonlinear equations.

No. of time-steps	*Solution*	p'	V_x	$V_y = V_z$	H_x'	$H_y' = H_z'$
20	1	0.5936	0.2052	0.1326	−0.2229	0.1576
	11	0.5912	0.2042	0.1326	−0.2167	0.1549
	111	0.5668	0.1848	0.1246	−0.1929	0.1538
50	1	−0.0383	0.3274	0.0832	−0.8583	0.6069
	11	−0.0397	0.3257	0.0834	−0.8535	0.6044
	111	−0.0425	0.3131	0.0852	−0.8197	0.5873
100	1	−0.1674	0.1534	−0.2860	0.1613	−0.1141
	11	−0.1668	0.1519	−0.2855	0.1600	−0.1135
	111	−0.1728	0.1534	−0.2915	0.1651	−0.1128

TABLE 2

Solution of Initial-Pressure Problem at $x = y = z = .4$. This is the degenerate case when $c_s^2 = 0$· 1 and 11 denote the exact and numerical solutions, respectively, of the linearized problem while 111 denotes the numerical solutions of the nonlinear equation.

No. of time-steps	*Solution*	p'	V_x	$V_y = V_z$	H_x'	$H_y' = H_z'$
20	1	0.6501	0.0	0.1595	−0.4852	0.0
	11	0.6462	−0.0003	0.1594	−0.4735	0.0
	111	0.6170	−0.0020	0.1493	−0.4374	0.0052
50	1	0.0828	0.0	0.1141	−1.9036	0.0
	11	0.0791	−0.0003	0.1141	−1.8904	0.0
	111	0.0684	−0.0016	0.1236	−1.8080	−0.0025
100	1	0.5384	0.0	−0.1823	−0.7646	0.0
	11	0.5344	−0.0003	−0.1821	−0.7524	−0.0002
	111	0.5693	0.0019	−0.1884	−0.8338	0.0029

TABLE 3

Solution of Initial-Pressure Problem at $x = y = z = .4$. This is the degenerate case when $c_f = b$, $-b$ and $c_s = b, -b$. 1 and 11 denote the exact and numerical solutions, respectively, of the linearized problem while 111 denotes the numerical solutions of the nonlinear equations.

No. of time-steps	*Solution*	p'	V_x	$V_y = V_z$	H_x'	$H_y' = H_z'$
20	1	0.7002	0.1789	0.0	0.0	0.0
	11	0.6991	0.1783	0.0	0.0	0.0
	111	0.6616	0.1591	−0.0008	0.0383	−0.0011
50	1	−0.1802	0.2700	0.0	0.0	0.0
	11	−0.1799	0.2694	0.0	0.0	−0.0001
	111	−0.1679	0.2478	0.0008	0.0029	−0.0018
100	1	−0.8505	−0.1057	0.0	0.0	0.0
	11	−0.8491	−0.1053	0.0	0.0002	0.0
	111	−0.8808	−0.1290	−0.0018	0.0299	0.0002

nonlinear system by noting that $H_x = H_0 + H_x'$, $H_y = H_y'$, $H_z = H_z'$, $p = p_0 + p'$, $\rho = \rho_0 + \rho'$ and putting the primed quantities and V_x, V_y, V_z in the matrices A_1, A_2, A_3 equal to zero.

In the numerical computation, the following initial conditions and values were used:

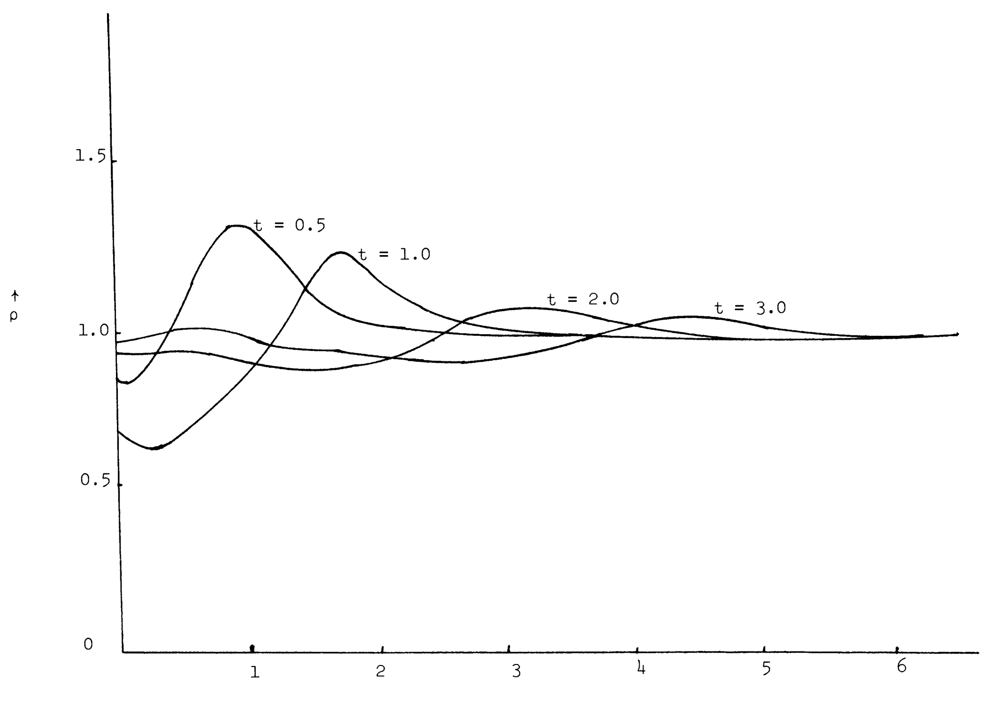

FIGURE 1. *Distribution of density in space at different times.*

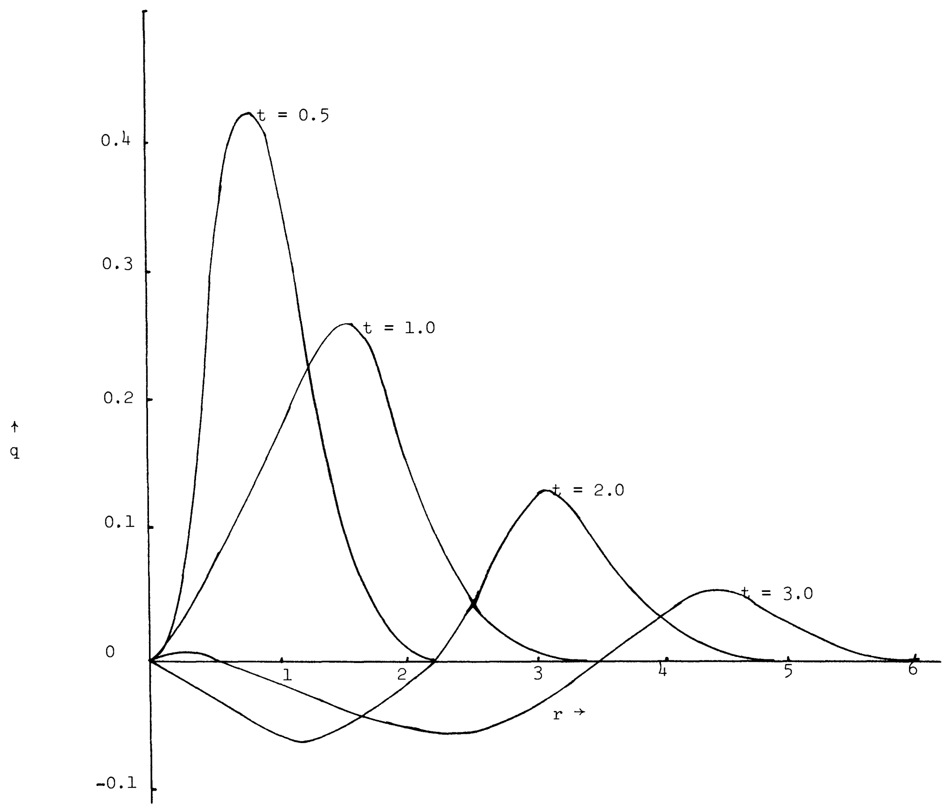

FIGURE 2. *Distribution of velocity in space at different times.*

$$p'(0, x, y, z) = \cos(x \cos \theta + y \sin \theta \cos \phi + z \sin \theta \sin \phi),$$

$$V(0, x, y, z) = H'(0, x, y, z) = 0,$$

with the following values for the different parameters of the problem: $H_0 = 5, \mu = 1, \rho_0 = 1, \theta = \phi = \pi/4, p = \rho^\gamma$ with $\gamma = 2$.

Computation was done for the region $0 \leqq x \leqq 1, 0 \leqq y \leqq 1, 0 \leqq z \leqq 1$. Boundary values were calculated by modifying the difference scheme as outlined in [5]. Mesh widths chosen are: $\Delta x = \Delta y = \Delta z = .1$, $\Delta t/\Delta x = .25$. The exact solution (1) and the numerical solution (11) for the linearized problem as well as the numerical solution (111) of the exact nonlinear equations are given in Table 1 for the point $x = y = z = .4$. The results for (11) are better than the ones reported in [4] because of finer meshes used here.

If we take $\theta = \pi/2$, then $c_s^2 = 0$ in this case [1] and we have a case where two bicharacteristic rays are taken more than once. The results for this case are given in Table 2.

If $c^2 = b^2$ and $\theta = 0$, then we have $c_f = b, -b$ and $c_s = b, -b$. For achieving such a case in this problem, we chose

$$\mu = 8\pi\rho_0/H_0^2 \quad \text{with } p = \rho^\gamma, \gamma = 2.$$

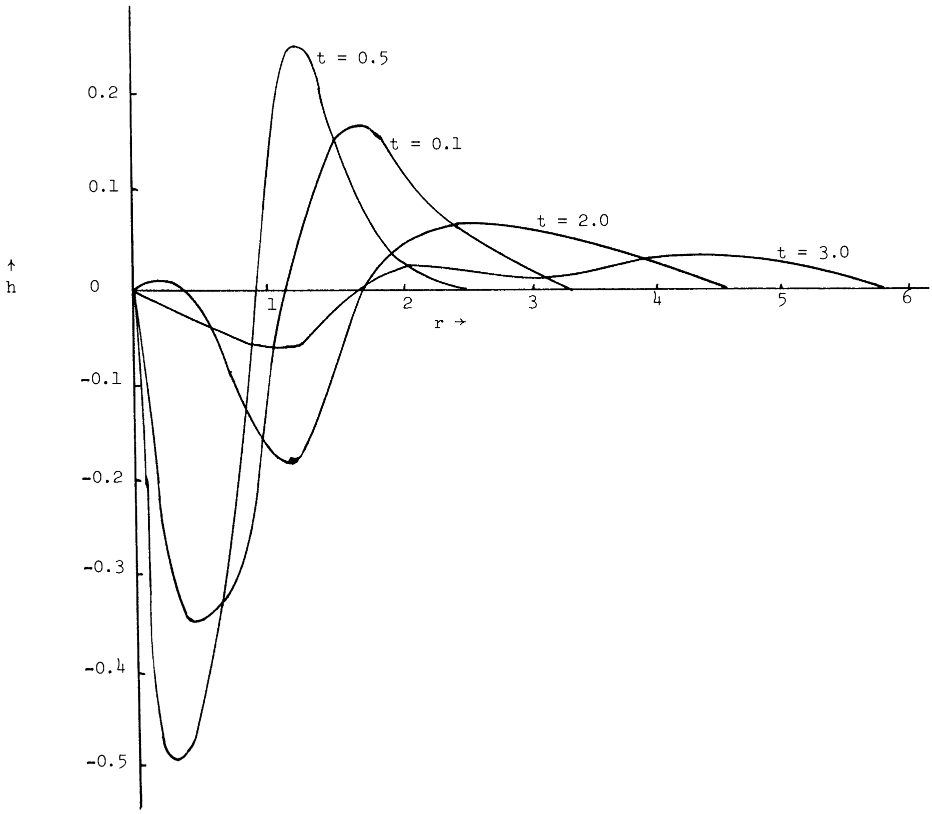

FIGURE 3. *Distribution of excess magnetic strength in space at different times* ($h = H - 1/r$).

The results for this case are given in Table 3.

(b) *Cylindrical Waves in Magnetohydrodynamics.* Assume an inviscid, conducting gas with a transverse magnetic field due to a thin current-carrying wire perpendicular to the plane of the fluid. In this problem, the velocity has only the radial component q while the magnetic field has only the transverse component H. Initially, the conducting fluid is assumed to be isothermal. The initial density distribution is assumed to be $\rho_0 = 1 + 2e^{-4r^2}$ where r is the radial distance and the initial magnetic field is of the form

$$H_0 \begin{array}{ll} = B/r, & \text{for } r \neq 0, \\ = 0, & \text{for } r = 0, \end{array}$$

where B is a constant. Briefly, the problem resembles that of the flow which results when compressed mass of conducting gas under a magnetic field, initially at rest, is suddenly released. The situation simulates, in simplified fashion, the conditions of a stellar explosion.

The equations governing the fluid are

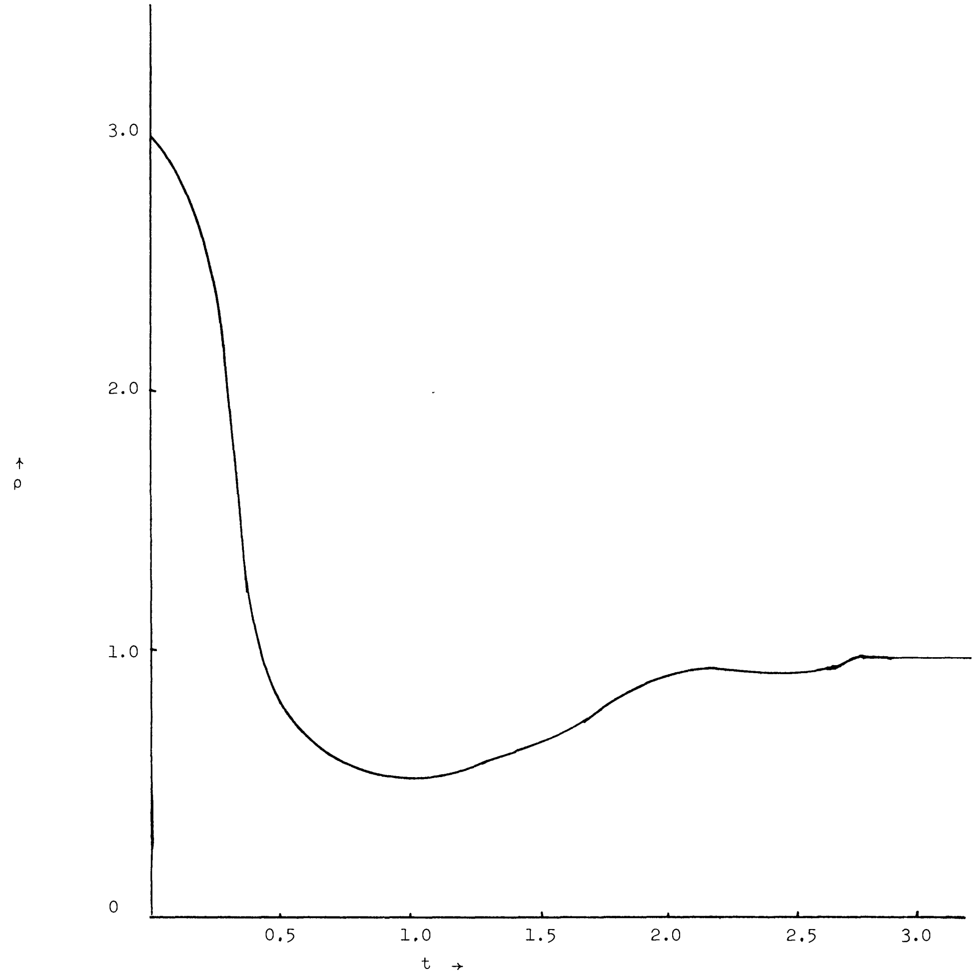

FIGURE 4. *Distribution of density at centre.*

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial r} + \frac{\mu H^2}{4\pi\rho r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\mu H}{4\pi\rho}\frac{\partial H}{\partial r},$$

$$\frac{\partial \rho}{\partial t} + q\frac{\partial \rho}{\partial r} + \rho\frac{\partial q}{\partial r} + \frac{\rho q}{r} = 0,$$

$$\frac{\partial H}{\partial t} + q\frac{\partial H}{\partial r} + H\frac{\partial q}{\partial r} + \frac{Hq}{r} = 0,$$

$$\left(\frac{\partial}{\partial t} + q\frac{\partial}{\partial r}\right)\left(\frac{p}{\rho^\gamma}\right) = 0.$$

At the centre $r = 0$, we have, from considerations of symmetry, $q = 0$, $\partial p/\partial r = 0$, $\partial \rho/\partial r = 0$. Also $H = 0$ at $r = 0$.

Numerical computations were performed with $\gamma = 1.4$, $B = 1$, $\mu = 1$, $\Delta r = .1$, $\Delta t/\Delta r = .1$ over 300 time intervals ($t = 3$). The results are shown in Figs. 1, 2, 3, 4.

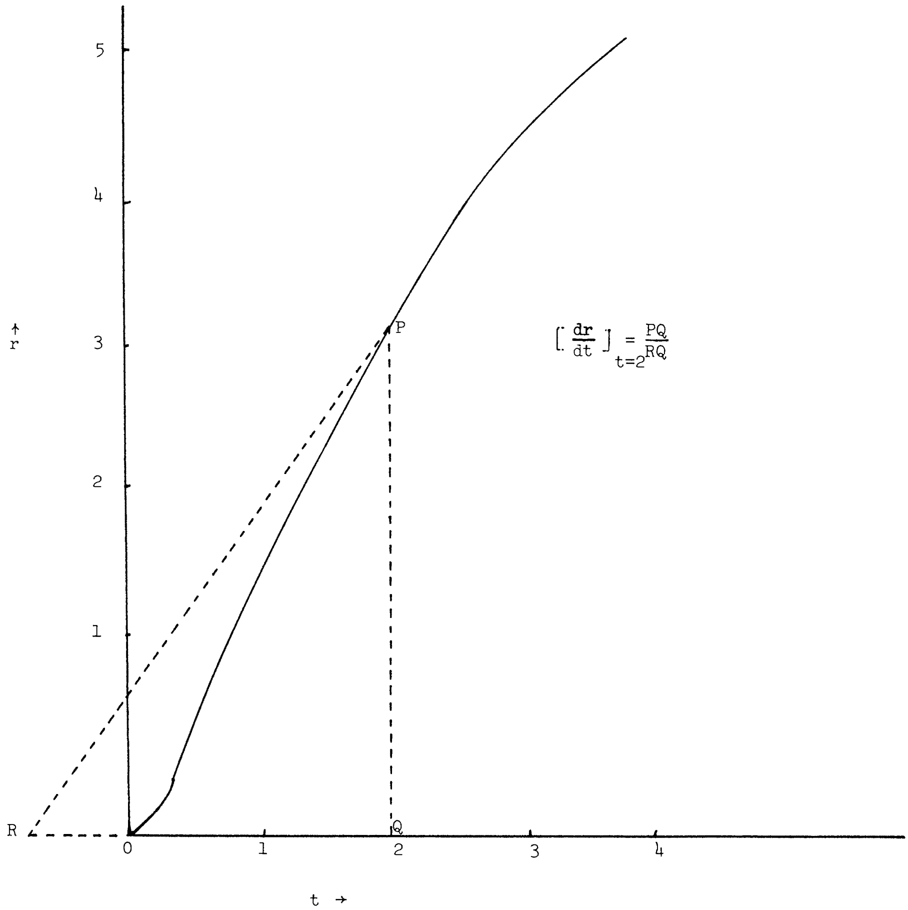

FIGURE 5. *Position of the wave-front at different times.*

Figure 5 gives the propagation of the wave front. The velocities of propagation of the wave front calculated from this curve at different times agree with the velocities calculated from the formula

$$dr/dt - q = (c^2 + H^2/4)^{1/2}$$

where c is, as usual, the local speed of sound.

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