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Reduction Formulas for Multiple Series

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Abstract. A simple procedure is given for reducing broad classes of multiple series to single series. Examples are given for double series.

Suppose that $A_1A_2 = A_3$, where A_i is a function of u and possesses a series expansion $A_i = \sum_n \phi_i(n, u)$. Then we have

$$(1) \qquad \sum_{m,n} \phi_1(n, u)\phi_2(m, u) = \sum_n \phi_3(n, u).$$

If both sides of (1) are multiplied by some function $f(u)$ and integrated over u, we shall have formally

$$(2) \qquad \sum_{m,n} F_1(m, n) = \sum_n F_2(n).$$

This rather trivial procedure can lead to some remarkable and useful results, as we shall illustrate by some examples.

If f and g are two analytic functions, then, upon multiplication of their Taylor series, we obtain

$$(3) \qquad \sum_{m,n=0}^{\infty} \frac{f^{(n)}(0)g^{(m)}(0)}{m!\,n!} F(m + n + 1) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} F(n + 1),$$

where F is any Mellin transform.

From the theory of elliptic functions [1], we have the Fourier series

$$(4) \quad \begin{aligned} &\text{(a)} \quad \operatorname{cn}(2Kx/\pi) = (2\pi/kK) \sum_0^{\infty} q^{(n+1/2)}(1 + q^{2n+1})^{-1} \cos(2n + 1)x, \\ &\text{(b)} \quad (2K/\pi) \operatorname{dn}(2Kx/\pi) = 1 + 4 \sum_1^{\infty} q^n(1 + q^{2n})^{-1} \cos 2nx, \\ &\text{(c)} \quad (2K/\pi) \operatorname{cn}(2Kx/\pi) \operatorname{dn}(2Kx/\pi) \\ &\qquad = (2\pi/kK) \sum_0^{\infty} (2n + 1)q^{n+1/2}(1 - q^{2n+1})^{-1} \cos(2n + 1)x, \end{aligned}$$

where $q = e^{-\pi K'/K}$. If we now let $K'/K = (2u/\pi)$, we find on multiplication of (4(a)) by (4(b)) that

$$(5) \qquad \sum_{m,n=-\infty}^{\infty} \frac{\cos(2m + 2n + 1)x}{\cosh(2m + 1)u \cosh 2nu} = 2 \sum_{n=0}^{\infty} \frac{(2n + 1) \cos(2n + 1)x}{\sinh(2n + 1)u},$$

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where the addition theorem and evenness for the cosine have been used to simplify the left-hand side.

Next, we multiply both sides of (5) by some summable function $f(x)$, with cosine transform $F(y)$, and integrate over x between the limits 0 and ∞. Thus, we have

$$\sum_{m,n=-\infty}^{\infty} \frac{F(|2m+2n+1|)}{\cosh(2m+1)u \cosh 2nu} = 2\sum_{n=0}^{\infty} \frac{(2n+1)F(2n+1)}{\sinh(2n+1)u}. \tag{6}$$

This remarkable result is valid for any summable function $F(x)$.

For example, consider $F_k(x) = 1$ for $(2k + \frac{3}{2}) > |x|$, 0 otherwise. Denoting the sum on the left-hand side of (6) by S_k, we find that it can be written

$$S_k = s_0 + s_1 + \cdots + s_k, \tag{7}$$

where

$$s_k = 4\sum_{n=-\infty}^{\infty} [\cosh(4n+2k+1)u + \cosh(2k+1)u]^{-1}. \tag{8}$$

On the other hand, the sum on the right-hand side of (6) is finite and we have

$$S_k = 2[\operatorname{csch} u + 3 \operatorname{csch} 3u + \cdots + (2k+1)\operatorname{csch}(2k+1)u]. \tag{9}$$

Therefore, $s_k = S_k - S_{k-1} = 2(2k+1)\operatorname{csch}(2k+1)u$ and, hence,

$$\begin{aligned} &\sum_{n=1}^{\infty} [\cosh 2nu + \cosh(2k+1)u]^{-1} \\ &\quad = \tfrac{1}{2}[(2k+1)\operatorname{csch}(2k+1)u - \tfrac{1}{2}\operatorname{sech}^2(k+\tfrac{1}{2})u], \qquad k = 0, 1, 2, \cdots. \end{aligned} \tag{10}$$

In a similar way, we can derive

$$\sum_{m,n=-\infty}^{\infty} \frac{F(m+n+1) + F(m-n)}{\sinh(2m+1)u \cosh(2n+1)u} = 8\sum_{n=1}^{\infty} \frac{nF(n)}{\cosh(2nu)}, \tag{11}$$

$$\sum_{k,m,n=-\infty}^{\infty} \frac{F(k+m+n+1) + F(k-m+n)}{\cosh(2ku)\cosh(2m+1)u \sinh(2n+1)u} = 8\sum_{n=1}^{\infty} \frac{n^2F(n)}{\sinh(2nu)}, \tag{12}$$

where F is any sine transform (and hence odd).

Thus, taking $F(1) = -F(-1) = 1$, $F(n) = 0$, $n \neq 1$, in (12), we obtain the interesting double series

$$\sum_{k,n=-\infty}^{\infty} \frac{\sinh[2(k+n)+1]u}{\cosh 2ku \sinh(2n+1)u[\cosh 2(2k+2n+1)u + \cosh 4u]} = \operatorname{csch}^2 2u. \tag{13}$$

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1. E. T. Whittaker & G. N. Watson. *A Course of Modern Analysis*, 4th edition, Cambridge Univ. Press, New York, 1962, p. 510.