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Brun's Constant

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Abstract. This note reviews previous work and presents new numerical data and analytical development concerning a constant that arises in Brun's famous theorem about twin primes.

1. Introduction. This note began as a review of Karst's table [1] deposited in the Unpublished Mathematical Tables file of this journal and listed in the review section of this issue. This led us to review the whole subject and to compute our own table [2]. This note reviews both of these tables in detail and also has additional analysis, especially concerning Fröberg's attempt to improve upon the Hardy-Littlewood conjecture for twin primes.

Since different authors use slightly different series, we begin with explicit definitions. We define *Brun's constant* by

$$B = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \frac{1}{31} + \cdots \tag{1}$$

with the twin primes as the denominators and with 5 taken twice, since it occurs in two pairs. The partial sums of (1) are tabulated in both of the tables [1] and [2], and (1) is the way the series is written by Landau [3]. We conclude below that probably

$$B = 1.90218 \pm 2\cdot 10^{-5}. \tag{2}$$

In obtaining this approximation, we are assuming the truth of the Hardy-Littlewood conjecture. More on this later.

Selmer [4] computed

$$S = \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots, \tag{3}$$

with the first pair deleted, while Fröberg [5] takes 5 only once in his

$$F = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots. \tag{4}$$

In his preface [1], Karst also mentions

$$K = 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \cdots, \tag{5}$$

and, at one time at least, he conjectured that K "closely approximates" π. (More on *that* later.) One has the relations

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$$B = S + \frac{8}{15} = F + \frac{1}{5} = K - \frac{4}{3}.$$

We prefer B, since that is the way everyone counts the twins: (3, 5) is the first pair, (5, 7) the second, etc., but we must admit that Brun himself [6] writes $1/5 + 1/7 + 1/11 + 1/13 + \cdots$, since it is convenient in the analysis to confine oneself to the twins $6N \mp 1$.

Karst's table [1] is a continuation—to the pair (393077, 393079)—of his two earlier tables previously reviewed [7], [8]. It not only has an unduly bulky foımat for the very limited range covered but also has numerous computational errors. We now find that his earlier tables [7], [8] also had errors. We discuss all this below.

An examination of these Karst tables motivated us to review the whole subject and to compute our own table. We had the first two million primes on tape—to $p = 32452843$. With this, and a trivial program, about 30 seconds computer time on a CDC 6700 suffices to evaluate the partial sums of (1) to that limit, together with extrapolations to infinity by using the Hardy-Littlewood conjecture.

TABLE 1

p	$\pi(p)$	$\pi_2(p)$	Sum of Inverse Twins	First-Order Extrapolation
1299709	100000	10250	1.71442 77999 16	1.90200 50649 40
2750159	200000	19462	1.72403 60977 15	1.90213 12696 54
4256233	300000	28349	1.72919 49994 11	1.90219 45682 57
5800079	400000	36826	1.73259 46834 13	1.90215 62627 21
7368787	500000	45204	1.73515 40809 43	1.90214 87517 94
8960453	600000	53661	1.73723 37280 71	1.90218 82887 32
10570841	700000	61885	1.73892 22762 30	1.90219 11615 64
12195257	800000	69967	1.74034 52270 11	1.90218 37298 27
13834103	900000	77975	1.74157 76245 86	1.90217 50764 34
15485863	1000000	86027	1.74267 74640 33	1.90218 07793 33
17144489	1100000	93998	1.74365 55695 02	1.90218 45702 07
18815231	1200000	101932	1.74453 87765 09	1.90218 76910 55
20495843	1300000	109744	1.74533 42233 45	1.90218 20017 80
22182343	1400000	117522	1.74606 38123 77	1.90217 83478 16
23879519	1500000	125358	1.74674 46172 76	1.90218 16679 53
25582153	1600000	133103	1.74737 13760 33	1.90218 08082 54
27290279	1700000	140815	1.74795 51905 44	1.90218 02181 25
29005541	1800000	148474	1.74849 94301 49	1.90217 73476 84
30723761	1900000	156143	1.74901 30783 75	1.90217 80145 38
32452843	2000000	163766	1.74949 59128 17	1.90217 59747 86

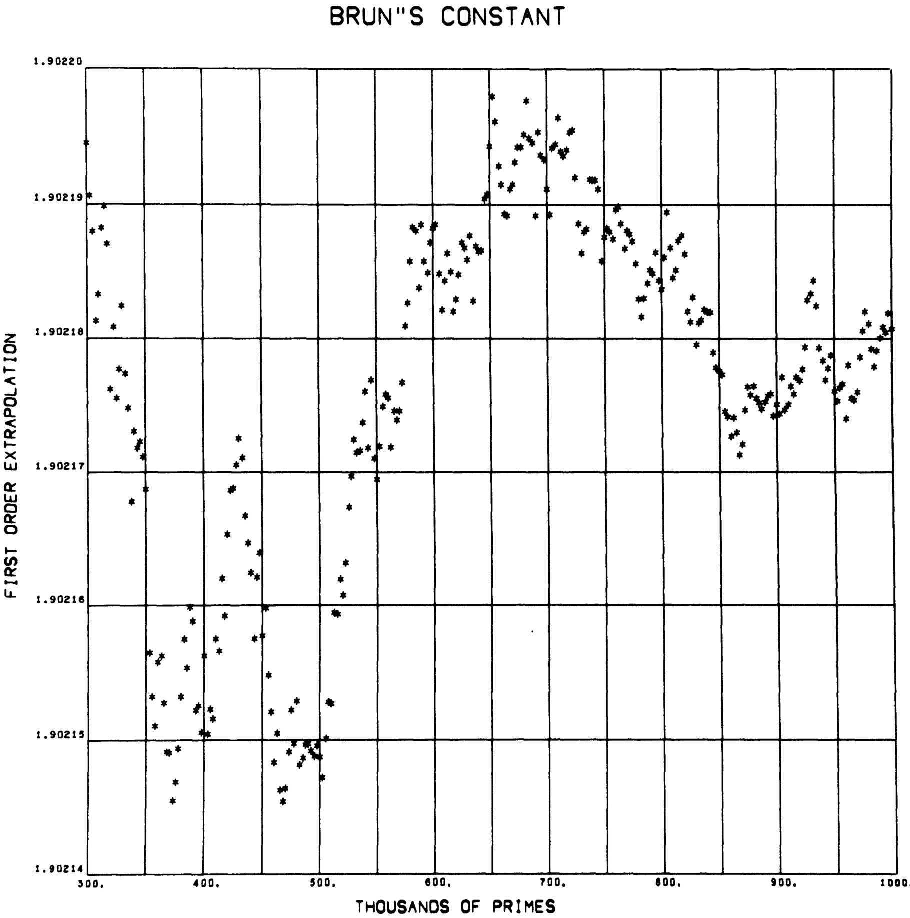

FIGURE 1

2. The New Table. Our "Brun's Constant" table [2] deposited in the UMT file lists, for

$$\pi(p) = 500(500)2 \cdot 10^6, \tag{6}$$

the following quantities:

p, $\pi(p)$, $\pi_2(p)$, Sum, First Order Extrapolation.

Here, p is the $\pi(p)$th prime and $\pi_2(p)$ is the number of prime-pairs $(q, q + 2)$ for $3 \leqq q \leqq p$. That is, the count includes $(p, p + 2)$, if that is a pair, as does Weintraub's table [9]. "Sum" is the partial sum of (1)—again including $1/p + 1/(p + 2)$ if $(p, p + 2)$ is a pair. "First Order Extrapolation" is this "Sum" increased by

$$\frac{4c_2}{\log p} = 2c_2 \int_p^\infty \frac{2\,dx}{x \log^2 x}, \tag{7}$$

where c_2 is the twin-prime constant [10].

In Table 1 we include 1/200th of the deposited table:

$$\pi(p) = 10^5(10^5)2 \cdot 10^6. \tag{8}$$

As in the original table, the last two columns are truncated to 12D from the double-precision 28D computations. Figure 1, which shows the First Order Extrapolation values for

$$\pi(p) = 3 \cdot 10^5(2500)10^6, \tag{9}$$

was plotted on a SC 4020.

Some comments on Figure 1. As $\pi(p)$ increases from $3 \cdot 10^5$ to 10^6, the partial sum of (1) increases from

$$1.729195 \quad \text{to} \quad 1.742677,$$

while the extrapolation, as shown, is confined within the interval (1.90214, 1.90220). This shows that the Hardy-Littlewood estimate is very accurate in this region. Relatively rapid changes correlate, of course, with the expected fluctuations. For example, there are only 183 prime-pairs between $\pi(5023307) = 350000$ and $\pi(5061919) = 352500$ instead of the expected 214 pairs; thus the abrupt drop of about $62/5 \cdot 10^6$ seen in Figure 1 at abcissa 350.

Figure 2 shows the continuation for $\pi(p) = 10^6(4000)2 \cdot 10^6$ with the same vertical

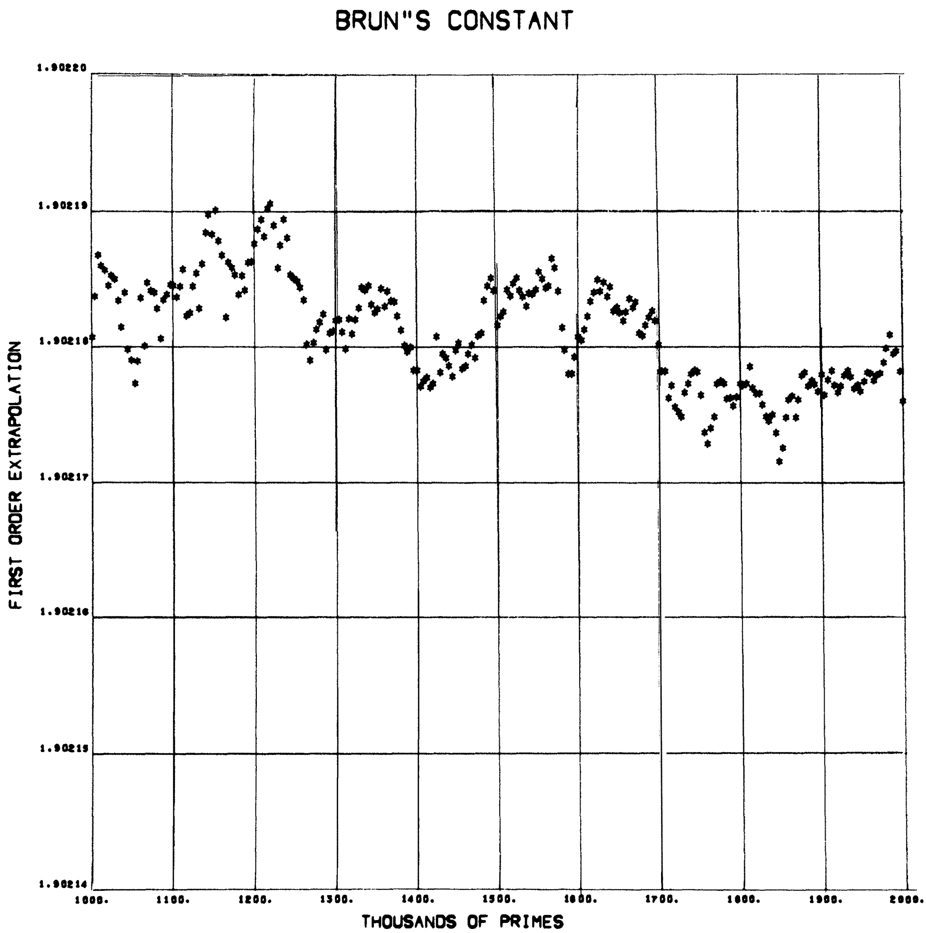

FIGURE 2

scale. Its ordinates are now much more stable, being confined between 1.902171 and 1.902191. Our best estimate for B is given in (2). We repeat: this estimate assumes the Hardy-Littlewood conjecture.

3. Fröberg's Modification. In [5], Fröberg attempts to improve upon (7) by replacing $1/\log^2 x$ by $\{P'(x)\}^2$ where $P(x)$ is the Riemann-Gram formula for $\pi(x)$:

$$P(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n \cdot n! \cdot \zeta(n+1)}. \tag{10}$$

The squared series $\{P'(x)\}^2$ is very complicated, and this leads Fröberg to an elaborate and lengthy computation. Frankly, we are not convinced that this change improves (7), for a number of reasons.

A. There is no convincing heuristic argument that this change gives a more accurate estimate of the number of twin primes.

B. It is well known in approximation theory that $f(x) \approx F(x)$ does not imply $f'(x) \approx F'(x)$.

C. Since all twins after (3, 5) are $6N \mp 1$, one should take into account the fact that the number of primes $6N - 1$ is usually closely given by $\frac{1}{2}$ li$(6N)$, while the number of $6N + 1$ primes is usually closer to the smaller $P(6N) - \frac{1}{2}$ li$(6N)$; cf. [11]. Thus, if any change in $1/\log^2 x$ is wanted, it could be argued that

$$2P'(x)/\log x - 1/\log^2 x \tag{11}$$

would be better than $\{P'(x)\}^2$. (It would also be *much* easier to compute.)

D. Any such change in $1/\log^2 x$ leads to a change in (7) that is dominated by the fluctuations that are seen in Figure 1.

And since Fröberg lists his sum for only a few scattered values of p, the resulting change in (7) is nullified by these random fluctuations. Fröberg, in fact, goes to $p < 2^{20} = 1048576$ and concludes that the F of (4) equals $1.70195 \pm 3 \cdot 10^{-5}$. We believe that this is too small. We return to (11) below.

4. Karst's Table. Karst's third installment [1] comprises the 1250 twin pairs from (239429, 239431) to (393077, 393079). The 2500 reciprocals and 2500 partial sums are printed on 207 11-inch × 15-inch computer sheets. This immense bulkiness for this limited range of data is attained by printing about 13 reciprocals and 13 partial sums along the right-hand edge of these computer sheets while the 10 inches on the left are left blank. (Would that Fermat had had such margins!)

We compared one of Karst's partial sums with our sum in [2] at $\pi(p) = 33000$, $p = 389171$, and found this discrepancy:

Shanks-Neild	1.6968620669 1614459837
Karst	1.6968560412 7032377889.

The Karst table purports to be accurate to 20D, but here it is only accurate to 5D. With some labor, we analyzed his errors:

A. The prime-pair 331908 ± 1 was omitted. In his preface, Karst indicates that at 400000 he has 3803 pairs, while Fröberg (presumably incorrectly) had 3804. We agree with 3804, as does Gruenberger's table [12].

B. Starting with $p = 21059$ in the first installment [7], there are curious division errors:

$$(1 + 10^{-8})/p \quad \text{instead of} \quad 1/p$$

for $p = 21059, 22963, 23743$, etc.,

$$(1 + 10^{-12})/p \quad \text{instead of} \quad 1/p$$

for $p = 22367, 23057, 23293$, etc., and even

$$(1 + 4 \cdot 10^{-8})/p \quad \text{instead of} \quad 1/p$$

at $p = 389299$. Thus, even before the vanished pair 331908 ± 1 the tables do not have the claimed accuracy.

Finally, we comment on Karst's π conjecture. By (2), we say that the K of (5) equals $3.23551 \pm 2 \cdot 10^{-5}$ if the Hardy-Littlewood conjecture is true. Since Karst's final partial sum in [1] is 1.69704, his partial sum for (5) is 3.03037, and he conjectures (or he did) that K may equal π. Viggo Brun himself expressed extreme skepticism in a letter to Karst: "Ich halte es für ausserordentlich unwahrscheinlich das 'meine' Konstante etwas mit π zu thun hat." Karst then wrote us, "Anyway, if Fröberg's computed result is correct, sometime in the future I will reach π."

Here is our estimate: That should occur at a prime p such that

$$3.23551 - 3.14159 \approx 4c_2/\ln p.$$

Thus, $p \approx 1.62 \cdot 10^{12}$. At this p, $\pi_2(p) \approx 2.93 \cdot 10^9$, and since Karst covers 1250 pairs in each installment, we think he will reach π in (or near) his 2,340,000th volume. Provided, of course, that he does not lose too many more prime-pairs, and that he corrects that mysterious division routine.

5. Brun's Constant. Though we refer to the last column in Table 1 as the "*First Order* Extrapolation," we must admit that we know of no higher-order approximation that would enable us to compute a more accurate value of B. Consider (11) rewritten as

$$\frac{2P'(x)}{\log x} - \frac{1}{\log^2 x} = \frac{1}{\log^2 x} - \frac{2}{x \log^2 x}\left(1 + \sum_{n=1}^{\infty} \frac{\log^n x}{n!} \frac{\zeta(n+1) - 1}{\zeta(n+1)}\right). \tag{12}$$

Since

$$\frac{\zeta(n+1) - 1}{\zeta(n+1)} \sim \frac{1}{2^{n+1}}, \tag{13}$$

one sees that the change in (7) obtained by replacing $1/\log^2 x$ in the integrand by (12) is a small one, since the right side of (12) is very close to

$$\frac{1}{\log^2 x} - \frac{1}{\sqrt{x} \log^2 x}.$$

We have accurately computed the change in the final entry in Table 1 brought about by such a replacement. Instead of the first-order extrapolation 1.90217597 shown there for $\pi(p) = 2 \cdot 10^6$, we now have a "second-order extrapolation" 1.90217334. Since this change is rather small compared with the fluctuations seen in Figure 2, it is unlikely that any such modification in (7) would really give B more accurately. (We should note that our second-order (11) and Fröberg's $\{P'(x)\}^2$ differ only by a very small third-order quantity. Thus, his second-order term would not alter this difficulty, and, as already stated, our (11) is much easier to compute.)

What is wanted, of course, is an analytic formulation of the fluctuations. As is known, cf. [13], the analogous $\pi(x) - P(x)$ can be computed, but not easily, with the complex zeros of the zeta function. We know of nothing similar for the twin primes. Lacking this, we have no assurance that the bounds $\pm 2 \cdot 10^{-5}$ in (2) are more than probably true.

To conclude, while B is a well-defined real number, its computation to 8 or 9 decimals seems to us extremely difficult, while 20 decimals is really impossible at the present time. Such a computation, if rigorous, would certainly entail a proof of the Hardy-Littlewood conjecture, and more besides.

However, one could easily go beyond our $\pi_2(32452843) = 163766$ prime-pairs. For example, Brent [14] has determined that

$$\pi_2(10^9) = 3424506.$$

Note added in proof. While this note was in process of publication, we learned of Bohman's work which subsequently appeared in "Some computational results regarding the prime numbers below 2,000,000,000," *BIT*, v. 13, 1973, pp. 242–244. Bohman goes to $p < 2 \cdot 10^9$ and he gives $F = 1.7021532$ there, using Fröberg's extrapolation. However, there are errors in his values of $\pi_2(x)$ at $x = 10^9$ and $x = 2 \cdot 10^9$, and perhaps some truncation error in his sum. We initiated a three-way correspondence, and he and Brent now agree that Brent's $\pi_2(10^9)$ was the correct count. Brent then continued to $1.25 \cdot 10^{10}$. At 10^9, Brent gives $B = 1.902160239321$.

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