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Iterates of the Unitary Totient Function*

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Abstract. The iterates of the unitary analogue of Euler's totient function $\varphi^*(n)$ are investigated empirically for $n \leqq 10^5$.

Introduction. Very little work has been done on the unitary analogue of arithmetic functions, except that of Hagis, Jr. [1] who investigated unitary amicable numbers. The divisor d is called unitary divisor of n if $(d, n/d) = 1$.

In this brief note, we investigate the iterations of the unitary analogue of Euler's totient function $\varphi^*(n)$. This function may be defined as follows:

$$\varphi^*(n) = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_s^{\alpha_s} - 1) \tag{1}$$

if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

From (1) it follows that $\varphi^*(n)$ is a multiplicative function. The rth iterate is denoted by

$$\varphi_r^*(n) = \varphi_1^*[\varphi_{r-1}^*(n)], \qquad r > 1 \quad \text{and} \quad \varphi_1^*(n) = \varphi^*(n). \tag{2}$$

Let $r = r(n)$ be the smallest integer such that $\varphi_r^*(n) = 1$. Erdös and Subbarao [2] stated that no nontrivial estimate for $r(n)$ exists and mentioned that probably $r(n) = o(n^\epsilon)$ for every $\epsilon > 0$. It is hoped that the numerical information provided here will be helpful for understanding some of the problems related to $\varphi_r^*(n)$.

Results. For $n \leqq 10^5$, we computed the maximum and minimum values of n for a given $r(n)$ such that $\varphi_r^*(n) = 1$. These values of n, along with the frequencies with which $r(n)$ appears, are presented in Table 1. Using the minimum values of n for $r(n) \leqq 27$ and the maximum values of n for $r(n) \leqq 15$, upper and lower bounds for $r(n)$ were estimated. These bounds are

$$\ln n < r(n) < 2.5 \ln n, \qquad n \leqq 10^5. \tag{3}$$

It is of interest to compare $r(n)$ with the corresponding $r(n)$ for Euler's totient function $\varphi(n)$ such that $\varphi_r(n) = 1$. S. S. Pillai [3] proved that

$$\ln (n/2)/\ln 3 + 1 \leqq r(n) \leqq \ln n/\ln 2 + 1. \tag{4}$$

Comparison of (3) and (4) shows that the lower bounds are very close. However, the upper bound in (3) is somewhat larger. Furthermore, the result (3) suggests an answer to a question raised in [2] that $r(n) < c \ln n$ has infinitely many solutions for some $c > 0$.

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TABLE 1

Minimum and maximum values of n such that $\varphi_r^*(n) = 1$, $n \leqq 10^5$.

r	Min n	Max n	*Frequency*
1	2	2	1
2	3	6	2
3	4	14	4
4	5	42	9
5	9	86	21
6	16	186	38
7	17	462	82
8	41	930	164
9	83	1986	261
10	113	4170	424
11	137	6510	749
12	257	14682	1097
13	773	29366	1721
14	977	50342	2592
15	1657	73410	4351
16	2048	99878	7299
17	2313	99890	10267
18	4001	99996	13829
19	5725	100000	16327
20	7129	99989	15989
21	11117	99987	12165
22	17279	99999	7368
23	19897	99997	3197
24	22409	99992	1465
25	39283	99936	458
26	43657	99867	95
27	55457	98713	24

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