MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 125, JANUARY, 1974

Four Large Amicable Pairs

By H. J. J. te Riele

Abstract. This note gives a report of systematic computer tests of Euler's rule and several Thabit-ibn-Kurrah-rules, in search of large amicable pairs. The tests have yielded four amicable pairs, which are much larger than the largest amicable pair thus far known.

1. The pair of 25-digit numbers

(45222 6553454520 8537974785, 45398 0132623392 8286140415)

has been the largest known amicable pair since 1946 ([8], [10]). This note gives four new amicable pairs with 32-, 40-, 81-, and 152-digit numbers, as a result of systematic computer tests by Euler's rule (Section 2) and several Thabit-ibn-Kurrah-rules (Sections 3 and 4).

In this research, primality of very large numbers N had to be established, where $N + 1$ can be easily factorized; this was done by use of the following:

THEOREM (LUCAS-LEHMER [11, p. 442]). *Let P and Q be relatively prime integers and let $U_0 = 0$, $U_1 = 1$, $U_{i+1} = PU_i - QU_{i-1}$ for $i \geqq 1$. If N is a natural number, relatively prime to $2P - 8Q$, and if $U_{N+1} \bmod N = 0$, while $U_{(N+1)/p} \bmod N \neq 0$ for each prime p dividing $N + 1$, then N is prime.*

It is convenient to choose $P = 1$, while Q has to be chosen such that $D^{(N-1)/2} \bmod N = -1$, where $D = P - 4Q$.

In the sequel, the indication "($Q = A$)" after a number means that primality of that number was established by use of this Lucas-Lehmer theorem, with $Q = A$. The computations were carried out on the Electrologica-X8 computer of the Mathematical Centre; the value of $U_i \bmod N$ was computed in $O(\log i)$ steps by use of the binary method (see [6, p. 360 (Exercise 15) and p. 421 (Exercise 26)]).

2. Euler's rule [4] for amicable numbers is given by: *$2^n pq$ and $2^n r$ are amicable numbers, if the three integers $p = 2^{n-m}f - 1$, $q = 2^n f - 1$ and $r = 2^{2n-m}f^2 - 1$ are primes, with $f = 2^m + 1$ and $n > m \geqq 1$.* For $m = 1$, this rule is due to Thabit ibn Kurrah and yields amicable numbers for $n = 2, 4, 7$, but for no other value $n \leqq 1000$ (see [12, p. 874]).* Only one more solution of Euler's rule was known thus far, viz., $m = 7$, $n = 8$ (Legendre, Chebyshev).

A systematic computer search for triples (p, q, r) such that both p, q and r are primes was carried out for all values of n, m with $n > m > 1$ and $r < 10^{132}$; this search yielded just one new solution, viz., $m = 11$, $n = 40$. Thus we have the new 40-digit amicable numbers:

Received November 30, 1972.

AMS (MOS) subject classifications (1970). Primary 10A40; Secondary 10A25, 10–04.

Key words and phrases. Amicable numbers, Lucas-Lehmer test.

* This corrects an error in [2, p. 571, footnote]. W. Borho has asked me to point out here, that his quotation of a correct, private communication of E. J. Lee was incorrect.

$$m_1 = 2724918040\,3937065577\,8575224081\,9405848576 = 2^{40}pq,$$
$$m_2 = 2724918040\,3961848563\,0625803878\,7235905536 = 2^{40}r,$$

with

$$p = 2^{29}3\cdot 683 - 1 = 110\,0048498687 \quad (Q = -1),$$
$$q = 2^{40}3\cdot 683 - 1 = 225289\,9325313023 \quad (Q = -13),$$
$$r = 2^{69}3^2683^2 - 1 = 24782985\,2050580016\,6853312511\ (Q = -4)$$

and

$$m_1/m_2 \approx 1 - 2^{-40}.$$

3. Definition. A Thabit-ibn-Kurrah-rule or Thabit-rule

$$T(b_1, b_2, p, c_1X - 1, c_2X - 1),$$

with given natural numbers b_1, b_2, a prime p, and linear polynomials

$$c_1X - 1, c_2X - 1 \in Z[X]$$

is a statement of the form:

$p^nb_1(c_1p^n - 1)$ and $p^nb_2(c_2p^n - 1)$ are amicable numbers, if $q_i = c_ip^n - 1$ is prime and prime to b_i for $i = 1, 2$ ($n = 1, 2, \cdots$).

For a more general definition see [2].

Walter Borho [2] presents a list of fifteen Thabit-rules, which are constructed from those amicable numbers of the form au, as (with $(a, us) = 1$, s prime), for which $p = u + s + 1$ is prime. Table 1 presents another seven Thabit-rules, constructed in the same way; this completes the list of Thabit-rules which can be constructed from the (at least) 67 published ([8], [9], [10], [2]) amicable pairs of the form au, as with $(a, us) = 1$, s prime.

TABLE 1

Seven new Thabit-rules $T(au, a, p, (u + 1)X - 1, (u + 1)\sigma(u)X - 1)$ obtained from amicable pairs au, as (with $(a, us) = 1$, s prime) such that $p = u + s + 1$ is prime.

No.	a	u	$\sigma(u)$	p	obtained from pair no.
(i)	$3^27^213\cdot 19\cdot 29$	$41\cdot 173 = 7093$	7308	14401	(33) of [3]
(ii)	$3^45\cdot 11^271$	$709\cdot 2129 = 1509461$	1512300	3021761	(31) of [3]
(iii)	$3^27^211\cdot 19\cdot 43\cdot 89$	$293\cdot 22961 = 6727573$	6750828	13478401	(8) of [5] top of p. 168
(iv)	2^331	$17\cdot 107\cdot 4339 = 7892641$	8436960	16329601	(34) of [3]
(v)	2^8	$257\cdot 33023 = 8486911$	8520192	17007103	(17) of [3]
(vi)	$2^319\cdot 137$	$83\cdot 218651 = 18148033$	18366768	36514801	(2) of [7]
(vii)	2^7263	$4271\cdot 280883 = 1199651293$	1199936448	2399587741	(18) of [3]

In the fifteen Thabit-rules of Borho, and the seven, given here, the numbers $q_1 = (u + 1)p^n - 1$ and $q_2 = (u + 1)\sigma(u)p^n - 1$ were tested for primality, for all values of $n \geqq 1$ such that $q_2 < 10^{120}$. Both q_1 and q_2 appeared to be prime in only three cases; these cases, together with those of Borho and Lee (see [2]) are listed in

Table 2. Table 2 also mentions the discoverers of the amicable pairs from which the Thabit-rules were obtained.

TABLE 2

Five cases in which Thabit-rules yield amicable pairs.

Thabit-rule	*obtained from an amicable pair discovered by*	*value of n for which both q_1 and q_2 are primes*	*discovered by*
1 of [2]	Pythagoras (?)	2	Borho
6 of [2]	Euler	1	Lee
6 of [2]	Euler	19	te Riele
(i) of Table 1	Escott	8	te Riele
(ii) of Table 1	Escott	1	te Riele

Next follow the details of the three new amicable pairs. Thabit-rule 6 of [2], $n = 19$, yields the 152-digit amicable numbers:

$$m_1 = 86\ 2593766501\ 4359638769\ 0953818787\ 1666597148\ 4088835777\ 4281383581\ 6831022646\ 6591332953\ 3162256868\ 3649647747\ 2706738497\ 3129580885\ 3683841099\ 1321499127\ 6380031055$$
$$= 3^4 5\cdot 11\cdot 5281^{19} 29\cdot 89\cdot q_1,$$
$$m_2 = 90\ 2364653062\ 3313066515\ 5201592687\ 0786444130\ 4548569003\ 8961540360\ 5363719932\ 5828701918\ 5759580345\ 2747004992\ 7532312907\ 0333233826\ 7840675607\ 3892061566\ 6452384945$$
$$= 3^4 5\cdot 11\cdot 5281^{19} q_2,$$

with

$$q_1 = 2\cdot 1291\cdot 5281^{19} - 1 = 13917\ 5701888775\ 9763088555\ 3289918626\ 7927088632\ 5517442305\ 8328801872\ 3382689621 \quad (Q = -7),$$
$$q_2 = 2^3 3^3 5^2 1291\cdot 5281^{19} - 1 = 37577439\ 5099695136\ 0339099388\ 2780292340\ 3139307889\ 7094225748\ 7765055313\ 3261979399 \quad (Q = -4)$$

and $m_1/m_2 = .955926$.

Thabit-rule (i), $n = 8$, yields the 81-digit amicable numbers:

$$m_1 = 5\ 4392258330\ 0492702317\ 4526035140\ 9264518101\ 4270450011\ 0522977234\ 9031452833\ 4873070667$$
$$= 3^2 7^2 13\cdot 19\cdot 29\cdot 14401^8 41\cdot 173\cdot q_1,$$
$$m_2 = 5\ 6040973336\ 5289816514\ 3019352151\ 4014539485\ 6000068942\ 0342360436\ 3362262293\ 3462548533$$
$$= 3^2 7^2 13\cdot 19\cdot 29\cdot 14401^8 q_2,$$

with

$$q_1 = 2\cdot 3547\cdot 14401^8 - 1 = 13122977\ 5913527520\ 5302674846\ 4924755893 \quad (Q = -15),$$

$$q_2 = 2^3 3^2 7 \cdot 29 \cdot 3547 \cdot 14401^8 - 1$$
$$= 95902720237\ 6059120035\ 1947778167\ 0116073351\ (Q = -24)$$

and $m_1/m_2 = .970580$.

Thabit-rule (ii), $n = 1$, yields the 32-digit amicable numbers:

$$m_1 = 72\ 3874114476\ 8207595520\ 9400624355 = 3^4 5 \cdot 11^2 71 \cdot 3021761 \cdot 709 \cdot 2129 \cdot q_1,$$
$$m_2 = 72\ 5235579669\ 5217952865\ 9056738845 = 3^4 5 \cdot 11^2 71 \cdot 3021761 \cdot q_2,$$

with

$$q_1 = 2 \cdot 3^3 27953 \cdot 3021761 - 1 = 456\ 1233402581\ (Q = -3),$$
$$q_2 = 2^3 3^4 5^2 71^2 27953 \cdot 3021761 - 1 = 689795327\ 4724758599\ (Q = -3)$$

and $m_1/m_2 = .998123$.

Remark. The two previous examples, of size 81D and 152D, offer contrary evidence to the conjecture [1] that if there exists an infinity of amicable pairs (m_1, m_2) with $m_1 < m_2$, then $\lim_{m_1 \to \infty} m_1/m_2 = 1$.

4. Table 2 of [2] lists five more Thabit-rules, which differ slightly from the Thabit-rules mentioned above in Section 3. The numbers q_1 and q_2 occurring in these five Thabit-rules were also tested for primality, for all $n \geqq 1$ with $q_2 < 10^{120}$. The results were negative in the sense that no pairs (q_1, q_2) were found with both q_1 and q_2 prime.

Acknowledgement. I would like to thank Mr. E. J. Lee for his suggestion to investigate the non-Thabitean case of Euler's rule (Section 2).

Mathematical Centre
2e Boerhaavestraat 49
Amsterdam, The Netherlands

1. J. Alanen, O. Ore & J. Stemple, "Systematic computations on amicable numbers," *Math. Comp.*, v. 21, 1967, pp. 242–245. MR **36** #5058.

2. W. Borho, "On Thabit ibn Kurrah's formula for amicable numbers," *Math. Comp.*, v. 26, 1972, pp. 571–578.

3. E. B. Escott, "Amicable numbers," *Scripta Math.*, v. 12, 1946, pp. 61–72. MR **8**, 135.

4. L. Euler, *De Numeris Amicabilibus*, Leonhardi Euleri Opera Omnia, Teubner, Leipzig and Berlin, Ser. I, vol. 2, 1915, pp. 63–162.

5. M. García, "New amicable pairs," *Scripta Math.*, v. 23, 1957, pp. 167–171. MR **20** #5158.

6. D. E. Knuth, *The art of computer programming*. Vol. 2. *Seminumerical Algorithms*, Addison-Wesley, Reading, Mass., 1969. MR **44** #3531.

7. E. J. Lee, "Amicable numbers and the bilinear diophantine equation," *Math. Comp.*, v. 22, 1968, pp. 181–187. MR **37** #142.

8. E. J. Lee & J. S. Madachy, "The history and discovery of amicable numbers—part 1," *J. Recreational Math.*, v. 5, 1972, pp. 77–93.

9. E. J. Lee & J. S. Madachy, "The history and discovery of amicable numbers—part 2," *J. Recreational Math.*, v. 5, 1972, pp. 153–173.

10. E. J. Lee & J. S. Madachy, "The history and discovery of amicable numbers—part 3," *J. Recreational Math.*, v. 5, 1972, pp. 231–249.

→ D. H. Lehmer, "An extended theory of Lucas' functions," *Ann. of Math.*, v. 31, 1930, pp. 419–448.

12. H. Riesel, "Lucasian criteria for the primality of $N = h \cdot 2^n - 1$," *Math. Comp.*, v. 23, 1969, pp. 869–875. MR **41** #6773.