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Negative Integral Powers of a Bidiagonal Matrix

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Abstract. The elements of the inverse of a bidiagonal matrix have been expressed in a convenient form. The higher negative integral powers of the bidiagonal matrix exhibit an interesting property: the (ij)th element of the $(-m)$th power is equal to the product of the corresponding element of the inverse by a Wronski polynomial, viz., the complete symmetric function of degree $(m-1)$ of the diagonal elements, $d_i, d_{i+1}, \ldots, d_j$, of the inverse matrix.

1. Introduction. Positive integral powers of a bidiagonal matrix with a fixed diagonal element b and superdiagonal element 1, have been reported by Varga [1]. In the present note, we shall find the negative integral powers of a general $n \times n$ bidiagonal matrix B, having diagonal elements b_i, $i = 1, 2, \ldots, n$, and superdiagonal elements c_j, $j = 1, 2, \ldots, n-1$.

One may express $B = (I - \Gamma)D^{-1}$, where I is the identity matrix and D^{-1} a diagonal matrix composed of the diagonal elements of B. Γ is null, except for the elements $\Gamma_{i\,i+1} = -c_i/b_{i+1}$, for $i = 1, 2, \ldots, n-1$, on its first superdiagonal. The powers of Γ can easily be evaluated. In fact, the nonzero elements of Γ^m are given by

$$(\Gamma^m)_{i\,i+m} = \prod_{k=i}^{i+m-1} (-c_k/b_{k+1}), \quad \text{for } i = 1, 2, \ldots, n-m,$$

occurring only on the mth superdiagonal.

The inverse E_1 of B may be calculated either by the usual method of cofactors, or from the following expansions:

$$\begin{aligned} E_1 &= B^{-1} = D[I - \Gamma]^{-1} \\ &= D[I + \Gamma + \Gamma^2 + \Gamma^3 + \cdots + \Gamma^{n-1}]. \end{aligned}$$

The elements of E_1 may be written in a convenient form as:

$$\text{(1a)} \qquad e_1(i,j) = 0 \qquad \text{for } i > j,$$

$$\text{(1b)} \qquad = 1/b_j = d_j \qquad \text{for } i = j,$$

$$\text{(1c)} \qquad = d_i \prod_{k=i}^{j-1} (-c_k/b_{k+1}) \quad \text{for } i < j.$$

The inverse is upper triangular but is not necessarily bidiagonal.

2. Powers of the Inverse. The product of E_1 with itself is a matrix E_2, which is also upper triangular. Elements of E_2 are given by

$$\text{(2)} \qquad \begin{aligned} e_2(i,j) &= e_1(i,j) \sum_{k=i}^{j} [d_k] \qquad &\text{for } i \leqq j, \\ &= 0 &\text{for } i > j. \end{aligned}$$

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Result (2) may be generalized. In fact, the nth power of E_1 is an upper triangular matrix E_n where the (i,j)th element, for $i \leqq j$, is given by

$$(3) \qquad e_n(i,j) = e_1(i,j) \sum_{k_1=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{n-2}=i}^{k_{n-3}} \sum_{k_{n-1}=i}^{k_{n-2}} [d_{k_1} d_{k_2} d_{k_3} \cdots d_{k_{n-2}} d_{k_{n-1}}].$$

Proof. Let us assume that result (3) is true for $n = m$.

$$e_{m+1}(i,j) = \sum_{k_0=i}^{j} [e_m(i,k_0)e_1(k_0,j)],$$

the other terms in the summation for $1 \leqq k_0 \leqq i-1$ and $j+1 \leqq k_0 \leqq n$, are zero, as both $e_m(p,q)$ and $e_1(p,q)$ are zero for $p > q$.

By writing the expression for $e_m(i,k_0)$ from result (3), which is assumed to be valid for $n = m$, we have

$$e_{m+1}(i,j) = \sum_{k_0=i}^{j} [e_1(i,k_0)e_1(k_0,j)]$$

$$\cdot \sum_{k_1=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{m-2}=i}^{k_{m-3}} \sum_{k_{m-1}=i}^{k_{m-2}} [d_{k_1} d_{k_2} d_{k_3} \cdots d_{k_{m-2}} d_{k_{m-1}}].$$

The first summation is done by Eq. (2) and the expression reduces to

$$e_{m+1}(i,j) = e_1(i,j) \sum_{k_0=i}^{j} [d_{k_0}] \cdot \sum_{k_1=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{m-2}=i}^{k_{m-3}} \sum_{k_{m-1}=i}^{k_{m-2}} [d_{k_1} d_{k_2} d_{k_3} \cdots d_{k_{m-2}} d_{k_{m-1}}].$$

After grouping the summations together, we find that the result is true for $n = m + 1$.

It has already been found true for $n = 2$ in Eq. (2), and therefore, by mathematical induction, we have the proof.

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1. R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962, p. 14. MR **28** # 1725.