

## On Tikhonov's Method for Ill-Posed Problems\*

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**Abstract.** For Tikhonov's regularization of ill-posed linear integral equations, numerical accuracy is estimated by a modulus of convergence, for which upper and lower bounds are obtained. Applications are made to the backward heat equation, to harmonic continuation, and to numerical differentiation.

**1. Introduction.** Tikhonov's method applies to the integral equation

$$(1.1) \quad \int_0^1 k(x, y)u_0(y)dy = g_0(x) \quad (0 \leq x \leq 1).$$

The problem is to find  $u_0$  if  $k$  and  $g_0$  are known.

This problem is ill-posed if  $k$  is any measurable kernel. That follows from Riemann's lemma:

$$(1.2) \quad \int_0^1 k(x, y) \sin \lambda y \, dy \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty$$

which shows that a small change in the data,  $\delta g_0$ , may correspond to a large change in the answer,  $\delta u_0$ : for some large  $\lambda$ , let  $\delta g_0(x)$  equal the integral (1.2); this is a small data-change corresponding to the large answer-change  $\delta u_0(y) = \sin \lambda y$ .

Many ill-posed problems can be stated as integral equations. In this paper, we will discuss three ill-posed problems: the *backward heat equation*, *harmonic continuation*, and *differentiation*. All will be stated as integral equations. To each, we will apply Tikhonov's method, and we will find the *rate of convergence*. The rate of convergence gives an estimate of the accuracy of the computed solution.

**2. Tikhonov's Method.** We suppose that  $u_0(y)$  solves the integral equation (1.1). The kernel  $k(x, y)$  is known, and a function  $g(x)$  is known such that

$$(2.1) \quad \|g - g_0\| \leq \epsilon.$$

We will use the notation  $\|\varphi\|$  to mean the  $L_2$ -norm:

$$\|\varphi\| = \left( \int_0^1 \varphi^2(x) dx \right)^{1/2}$$

The assumption (2.1) is used instead of  $g = g_0$  because, in practical applications, the

Received September 25, 1973.

AMS (MOS) subject classifications (1970). Primary 65M30, 65R05; Secondary 65N99.

Key words and phrases. Ill-posed, improperly posed, regularization, Tikhonov's method, backward heat equation, harmonic continuation, numerical differentiation.

\* This work was supported by A.E.C. contract AT(04-3)-767, Project Agreement No. 12

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data are usually known only apart from certain errors of measurement. Even if  $g_0$  is known exactly, it is advisable to assume that there is some small data-error if any numerical computations are to be used; for numerical computations produce rounding errors, which have the same effect as exact computations based on inexact data. This point has been stressed by Wilkinson [4].

Tikhonov makes the assumption that  $u_0(y)$  satisfies an inequality

$$(2.2) \quad \Omega^2(u_0) < \infty$$

where  $\Omega^2(u)$  is a functional of the form

$$(2.3) \quad \Omega^2(u) = \int_0^1 \sum_{j=0}^p a_j(x) \left[ \frac{d^j u(x)}{dx^j} \right]^2 dx$$

where  $p \geq 1$  and where the functions  $a_j(x)$  are positive and continuous. For instance, if  $a_0 \equiv a_1 \equiv 1$  and if  $p = 1$ , we assume

$$(2.4) \quad \Omega^2(u_0) = \int_0^1 (u_0(x)^2 + u_0'(x)^2) dx < \infty.$$

Then, a number  $\alpha > 0$  is chosen, and a function  $u(x)$  approximating  $u_0(x)$  is computed by minimizing

$$(2.5) \quad \|Ku - g\|^2 + \alpha\Omega^2(u)$$

where

$$(2.6) \quad Ku = \int_0^1 k(x, y)u(y)dy.$$

It is assumed that

$$(2.7) \quad Ku = 0 \text{ only if } \|u\| = 0.$$

The parameter  $\alpha$  is related to the tolerance  $\epsilon$  in the inequality  $\|g - g_0\| \leq \epsilon$ .

It is assumed that

$$(2.8) \quad c_1\epsilon^2 \leq \alpha \leq c_2\epsilon^2$$

where  $c_1$  and  $c_2$  are positive numbers that are independent of  $\epsilon$ . The following theorem was proved by Tikhonov:

**THEOREM.** *Let (2.7) hold. Let*

$$(2.9) \quad Ku_0 = g_0, \text{ where } \Omega(u_0) < \infty.$$

*Let  $\|g_\epsilon - g_0\| \leq \epsilon$ , and let  $u = u_\epsilon$  minimize*

$$(2.10) \quad \|Ku - g_\epsilon\|^2 + \alpha\Omega^2(u)$$

*where  $\alpha$  satisfies (2.8). Then the functions  $u_\epsilon(x)$  converge uniformly to  $u_0(x)$  as  $\epsilon \rightarrow 0$ .*

*Proof.* Since we shall need the details of the proof of this simple theorem, we shall reproduce them here. Since  $u_\epsilon$  minimizes (2.10), we have

$$\begin{aligned} \|Ku_\epsilon - g_\epsilon\|^2 + \alpha\Omega^2(u_\epsilon) &\leq \|Ku_0 - g_\epsilon\|^2 + \alpha\Omega^2(u_0) \\ &\leq \epsilon^2 + \alpha\Omega^2(u_0) \\ &\leq \epsilon^2(1 + c_2\Omega^2(u_0)). \end{aligned}$$

It follows that

$$(2.11) \quad \Omega^2(u_\epsilon) \leq c_1^{-1} + \Omega^2(u_0)$$

and

$$(2.12) \quad \begin{aligned} \|Ku_\epsilon - g_0\| &\leq \|g_\epsilon - g_0\| + \|Ku_\epsilon - g_\epsilon\| \\ &\leq \epsilon[1 + (1 + c_2\Omega^2(u_0))^{1/2}]. \end{aligned}$$

The inequality (2.11) shows that the functions  $u_\epsilon(x)$  lie in the class of functions  $u(x)$  satisfying

$$(2.13) \quad \Omega^2(u) \leq \text{constant} = c_1^{-1} + \Omega^2(u_0).$$

This class is *compact* because the functions  $u$  satisfying (2.13) are equicontinuous and uniformly bounded. Therefore, there is a uniformly convergent subsequence of the functions  $u_\epsilon(x)$ .

It now suffices to prove that *every* uniformly convergent subsequence of the functions  $u_\epsilon(x)$  has the *same* limit,  $u_0(x)$ ; then the *whole* sequence  $u_\epsilon(x)$  must converge to  $u_0(x)$ . Now, if  $\varphi(x)$  is the uniform limit of a subsequence of the functions  $u_\epsilon(x)$ , the inequality (2.12) implies  $\|K\varphi - g_0\| = 0$  which says  $\|K(\varphi - u_0)\| = 0$  which implies  $\varphi = u_0$ , by (2.7).  $\square$

In this proof, it is assumed that  $u_\epsilon$  minimizes the functional (2.10) over the *entire* class of functions for which  $\Omega(u) < \infty$ . This assumption is not necessary. In some applications, one knows that the unknown lies in some *subset*,  $S$ , of the functions for which  $\Omega(u) < \infty$ . If  $u_\epsilon$  minimizes the functional only over the subset,  $S$ , the proof shows that the conclusion still holds:  $u_\epsilon(x)$  tends uniformly to  $u_0(x)$  as  $\epsilon \rightarrow 0$ .

The *existence* of a minimizing solution,  $u$ , depends on the prescribed subset,  $S$ , according to the calculus of variations; we will assume that  $u$  exists.

The *uniqueness* of the minimizing solution is established as follows: Assume the subset  $S$  is convex. Let  $u_1$  and  $u_2$  be minimizing functions. Then  $u = \frac{1}{2}(u_1 + u_2)$  is also minimizing, as we will prove by the parallelogram-laws:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

and

$$\Omega^2(p + q) + \Omega^2(p - q) = 2(\Omega^2(p) + \Omega^2(q)).$$

Let

$$x = \frac{1}{2}(Ku_1 - g), \quad y = \frac{1}{2}(Ku_2 - g),$$

$$p = \frac{1}{2}u_1, \quad q = \frac{1}{2}u_2.$$

Then, if  $u = \frac{1}{2}(u_1 + u_2)$ ,

$$\|Ku - g\|^2 + \|\frac{1}{2}K(u_1 - u_2)\|^2 = \frac{1}{2}(\|Ku_1 - g\|^2 + \|Ku_2 - g\|^2)$$

and

$$\Omega^2(u) + \Omega^2(\frac{1}{2}(u_1 - u_2)) = \frac{1}{2}(\Omega^2(u_1) + \Omega^2(u_2)).$$

Adding the first equation to  $\alpha$  times the second, we see that  $u$  is minimizing, and that

$$\|\frac{1}{2}K(u_1 - u_2)\|^2 + \alpha\Omega^2(\frac{1}{2}(u_1 - u_2)) = 0.$$

Therefore,  $u_1 = u_2$ .

**3. Regularization and Convergence.** The proof of Tikhonov's theorem relies on the principle of regularization: If

$$(3.1) \quad \|Ku_\epsilon - Ku_0\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and if the functions  $u_\epsilon(x)$  and  $u_0(x)$  satisfy some inequality

$$(3.2) \quad \Omega(u) \leq \omega_1$$

where  $\omega_1$  is independent of  $\epsilon$ , then  $u_\epsilon \rightarrow u_0$ . The inequality (3.2) may be called the regularization-inequality. Without it, the convergence  $u_\epsilon \rightarrow u_0$  would fail if the problem  $Ku_0 = g_0$  is ill-posed.

In order to determine the rate of convergence, it is useful to define a *modulus of regularization*,  $\rho(\epsilon)$ . Let  $\mu(u)$  be some norm defined for all functions  $u(x)$  in a linear space  $S$  of functions for which  $\Omega(u) < \infty$ . For instance, we could let

$$(3.3) \quad \mu(u) = \max_{0 \leq x \leq 1} |u(x)|$$

or

$$(3.4) \quad \mu(u) = \|u\|$$

or

$$(3.5) \quad \mu(u) = (\|u\|^2 + \dots + \|u^{(q)}\|^2)^{1/2},$$

where  $q$  is some integer less than the order  $p$ , which appears in the definition (2.3) of  $\Omega$ .

If the maximum-norm (3.3) is used, the uniform convergence of  $u_\epsilon(x)$  to  $u_0(x)$  is expressed by the statement:

$$(3.6) \quad \mu(u_\epsilon - u_0) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

If the  $L_2$ -norm (3.4) is used, then mean-square convergence is expressed by (3.6).

If the norm (3.5) is used, then mean-square convergence of  $u$  and its derivatives up to order  $q$  is expressed by (3.6).

We shall say that the functional  $\Omega$  regularizes the operator  $K$  with respect to the norm  $\mu$  if  $\mu(u_n) \rightarrow 0$  for every sequence of functions  $u_n(x)$  such that

$$(3.7) \quad \|Ku_n\| \rightarrow 0 \quad \text{while} \quad \Omega(u_n) \leq 1.$$

Under this condition we can define the *modulus of regularization*

$$(3.8) \quad \rho(\epsilon) \equiv \sup_{\|Ku\| \leq \epsilon; \Omega(u) \leq 1} \mu(u).$$

We note that  $\rho(\epsilon)$  tends monotonely to zero as  $\epsilon \rightarrow 0$ ; if this were not true, there would be some  $\delta > 0$  and a sequence of positive  $\epsilon_n$  tending to zero and a sequence of functions  $u_n(x)$  such that

$$\|Ku_n\| \leq \epsilon_n, \quad \Omega(u_n) \leq 1, \quad \mu(u_n) \geq \delta.$$

But this is impossible, since there is a subsequence of the  $u_n$  whose  $\mu$ -norms tend to zero.

Since the functionals  $\|Ku\|$  and  $\Omega(u)$  are both homogeneous of degree 1, we have for every positive  $\omega$

$$(3.9) \quad \sup_{\|Ku\| \leq \epsilon; \Omega(u) \leq \omega} \mu(u) = \sup_{\|K(u/\omega)\| \leq \epsilon/\omega; \Omega(u/\omega) \leq 1} \omega\mu(u/\omega) = \omega\rho(\epsilon/\omega).$$

Next we define a modulus of convergence,  $\sigma(\epsilon, \alpha)$ , for Tikhonov's method. Let  $\alpha$  be a positive number. We first suppose  $\Omega(u_0) \leq 1$ . And we suppose that  $u$  minimizes the functional

$$(3.10) \quad \|Ku - g\|^2 + \alpha\Omega^2(u)$$

where  $\|g - Ku_0\| \leq \epsilon$ . Thus,  $u$  depends on  $g$ , which is related by an inequality to  $u_0$ . (Since the functional (3.10) is quadratic in  $u$ , it is easy to show that  $u$  is a *linear* transform of  $g$ ; but we do not require this information here.) Let us write

$$(3.11) \quad u = T_\alpha(g).$$

We now define the *modulus of convergence*

$$(3.12) \quad \sigma(\epsilon, \alpha) = \sup_{\|g - Ku_0\| \leq \epsilon; \Omega(u_0) \leq 1} \mu(T_\alpha g - u_0).$$

This definition takes account of the dependence of  $u$  on the parameter  $\alpha$ . The modulus  $\sigma(\epsilon, \alpha)$  measures, in terms of the norm  $\mu$ , the worst error obtainable when Tikhonov's method is used to approximate a solution  $u_0$  in the class  $\Omega(u_0) \leq 1$ , where the given  $g$  equals  $Ku_0$  apart from an error whose  $L_2$ -norm is at most  $\epsilon$ .

Suppose we know that  $\Omega(u_0) \leq \omega$  instead of  $\Omega(u_0) \leq 1$ . Suppose that  $\|g - Ku_0\| \leq \epsilon$ . Under these conditions, we have

$$(3.13) \quad \mu(u - u_0) \leq \omega\sigma(\epsilon/\omega, \alpha)$$

since the function  $v = u/\omega$  minimizes

$$(3.14) \quad \|Kv - g/\omega\|^2 + \alpha\Omega^2(v)$$

and since

$$\|g/\omega - K(u_0/\omega)\| \leq \epsilon/\omega \text{ and } \Omega(u_0/\omega) \leq 1.$$

Hence,  $\mu(u/\omega - u_0/\omega) \leq \sigma(\epsilon/\omega, \alpha)$  and (3.13) follows.

We have just proved, for all  $\omega > 0$ ,

$$(3.15) \quad \omega\sigma(\epsilon/\omega, \alpha) = \sup_{\|g - Ku_0\| \leq \epsilon; \Omega(u_0) \leq \omega} \mu(T_\alpha g - u_0).$$

This gives the rate of convergence for solutions in the class  $\Omega(u_0) \leq \omega$ . If we know that  $\Omega(u_0) \leq \omega$ , and if we know that  $\|g - Ku_0\| \leq \epsilon$ , the error in Tikhonov's method is at most equal to  $\omega\sigma(\epsilon/\omega, \alpha)$ . The following theorem shows how to estimate this quantity in terms of the modulus of regularization.

**THEOREM.** *The modulus of convergence,  $\sigma(\epsilon, \alpha)$ , is related to the modulus of regularization,  $\rho(\epsilon)$ , by the inequality*

$$(3.16) \quad \omega\rho(\epsilon/\omega) \leq \omega\sigma(\epsilon/\omega, \alpha) \leq \omega'\rho(\epsilon'/\omega')$$

where, if  $\lambda = \alpha\omega^2/\epsilon^2$ ,

$$(3.17) \quad \epsilon' = (1 + \sqrt{1 + \lambda})\epsilon, \quad \omega' = (1 + \sqrt{1 + 1/\lambda})\omega.$$

*Remark.* Since the lower bound  $\omega\rho(\epsilon/\omega)$  in (3.16) is independent of  $\alpha$ , this theorem suggests how to choose  $\alpha$  well if all that is known about  $u_0$  is that  $\|g - Ku_0\| \leq \epsilon$  and  $\Omega(u_0) \leq \omega$ : choose  $\alpha = \epsilon^2/\omega^2$ . This choice reduces (3.16) to the form

$$(3.18) \quad \omega\rho(\epsilon/\omega) \leq \omega\sigma(\epsilon/\omega, \alpha) \leq (1 + \sqrt{2})\omega\rho(\epsilon/\omega).$$

The two sides of this inequality are equal, apart from the numerical factor  $1 + \sqrt{2}$ .

*Proof of the theorem.* First we will prove

$$(3.19) \quad \omega\rho(\epsilon/\omega) \leq \omega\sigma(\epsilon/\omega, \alpha).$$

Given any number  $\theta < 1$ , and given any  $\epsilon > 0$ , we can find a function  $\varphi$  for which

$$\|K\varphi\| \leq \epsilon, \quad \Omega(\varphi) \leq 1, \quad \mu(\varphi) \geq \theta\rho(\epsilon).$$

Consider the problem of minimizing

$$\|Ku - g\|^2 + \alpha\Omega^2(u)$$

where

$$0 = g = K\varphi - K\varphi.$$

Note that, if we set  $u_0 = \varphi$ , we have

$$\|g - Ku_0\| \leq \epsilon \text{ and } \Omega(u_0) \leq 1.$$

Therefore a solution  $u$  of the minimum problem must satisfy

$$\mu(u - u_0) \leq \sigma(\epsilon, \alpha).$$

But  $u = 0$  satisfies the minimum problem, since  $g = 0$ . The last inequality now becomes  $\mu(\varphi) \leq \sigma(\epsilon, \alpha)$ . But  $\mu(\varphi) \geq \theta\rho(\epsilon)$ . Therefore,  $\theta\rho(\epsilon) \leq \sigma(\epsilon, \alpha)$ . Since this

holds for every  $\theta < 1$ , the inequality (3.19) follows.

Now we will prove the right-hand side of (3.16). Let  $\theta < 1$ . If  $\epsilon$  and  $\omega$  are any positive numbers, Eq. (3.15) implies that there are functions  $u_0$  and  $g$  satisfying

$$(3.20) \quad \theta \omega \sigma(\epsilon/\omega, \alpha) \leq \mu(T_\alpha g - u_0)$$

where

$$\Omega(u_0) \leq \omega \quad \text{and} \quad \|g - Ku_0\| \leq \epsilon.$$

If  $u = T_\alpha g$ , then

$$\|Ku - g\|^2 + \alpha \Omega^2(u) \leq \epsilon^2 + \alpha \omega^2$$

since the right-hand side is obtained by substituting  $u_0$  for the minimizing function  $u$ . Now  $\|Ku - g\|^2$  and  $\alpha \Omega^2(u)$  are each  $\leq \epsilon^2 + \alpha \omega^2$ . Therefore,

$$\|Ku - g\| \leq (\epsilon^2 + \alpha \omega^2)^{1/2}$$

and

$$\Omega(u) \leq (\omega^2 + \epsilon^2/\alpha)^{1/2}.$$

Using the triangle inequality, we find

$$(3.21) \quad \begin{aligned} \|K(u - u_0)\| &\leq \|Ku - g\| + \|g - Ku_0\| \\ &\leq (\epsilon^2 + \alpha \omega^2)^{1/2} + \epsilon = \epsilon'. \end{aligned}$$

Since the functional  $\Omega$  also satisfies a triangle inequality, we have

$$(3.22) \quad \begin{aligned} \Omega(u - u_0) &\leq \Omega(u) + \Omega(u_0) \\ &\leq (\omega^2 + \epsilon^2/\alpha)^{1/2} + \omega = \omega'. \end{aligned}$$

Equation (3.9) now implies

$$(3.23) \quad \mu(u - u_0) \leq \omega' \rho(\epsilon'/\omega').$$

Equation (3.20) now yields

$$(3.24) \quad \theta \omega \sigma(\epsilon/\omega, \alpha) \leq \mu(u - u_0) \leq \omega' \rho(\epsilon'/\omega').$$

Letting  $\theta \rightarrow 1$ , we obtain the upper bound in (3.16).  $\square$

**4. Harmonic Continuation.** According to Poisson's formula, if a harmonic function equals  $u_0(\theta)$  on the unit circle, then it equals

$$(4.1) \quad Ku_0(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \theta_1) + r^2} u_0(\theta_1) d\theta_1$$

on an interior circle of radius  $r < 1$ . Let  $r$  be fixed. Then consider the problem of inverting Poisson's formula numerically.

We are given a function  $g(\theta)$  such that

$$(4.2) \quad \|Ku_0 - g\| \leq \epsilon.$$

The function  $u_0$  is unknown. It is supposed to be a periodic function satisfying some inequality

$$(4.3) \quad \Omega(u_0) \leq \omega.$$

For simplicity, suppose that  $\Omega^2(u)$  has the form

$$(4.4) \quad \Omega^2(u) = \int_0^{2\pi} [u(\theta)^2 + u'(\theta)^2] d\theta.$$

Applying Tikhonov's method, we choose a number  $\alpha > 0$  and state the problem of minimizing  $\|Ku - g\|^2 + \alpha\Omega^2(u)$  over the class,  $S$ , of periodic functions for which  $\Omega^2(u) < \infty$ . By periodic, we mean that  $u(0) = u(2\pi)$  and  $u'(0) = u'(2\pi)$ . With these boundary conditions, the calculus of variations shows that  $u$  must be the solution of the integro-differential equation

$$\alpha \left( -\frac{d^2u}{d\theta^2} + u \right) + K(Ku - g) = 0.$$

This equation can be solved numerically by finite-difference methods, although there is some difficulty if the parameter  $\alpha$  is very small.

Here we will only consider the dependence of the minimizing function  $u$  on the parameter  $\alpha$ : we will study the  $L_2$ -norm,

$$\mu(u - u_0) = \|u - u_0\|$$

as a function of  $\alpha$ .

By the discussion in the preceding section, we know that  $\|u - u_0\|$  is at most equal to  $\omega\sigma(\epsilon/\omega, \alpha)$  if the solution  $u_0$  satisfies the inequalities (4.2) and (4.3). Moreover, we have

$$(4.5) \quad \omega\rho(\epsilon/\omega) \leq \omega\sigma(\epsilon/\omega, \alpha) \leq \omega'\rho(\epsilon'/\omega')$$

according to (3.16), where

$$(4.6) \quad \rho(\epsilon) = \sup \|\varphi\| \text{ if } \|K\varphi\| \leq \epsilon \text{ and } \Omega(\varphi) \leq 1.$$

The inequality (4.5) holds for every  $\alpha$ , with  $\omega'$  and  $\epsilon'$  defined in (3.17). If  $\alpha = \epsilon^2/\omega^2$ , we may use the inequality (3.18). Because of the inequalities (3.16) and especially (3.18), we find it sufficient to estimate  $\rho(\epsilon)$  in order to estimate the error  $\|u - u_0\|$ .

Let  $\varphi(\theta)$  have the Fourier series

$$\varphi(\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta).$$

Then the inequality  $\Omega(\varphi) \leq 1$  implies

$$(4.7) \quad 2\pi A_0^2 + \pi \sum_{n=1}^{\infty} (A_n^2 + B_n^2) + \pi \sum_{n=1}^{\infty} n^2(A_n^2 + B_n^2) \leq 1.$$

For  $K\varphi$  we have the Fourier series

$$K\varphi(\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Therefore, the inequality  $\|K\varphi\| \leq \epsilon$  implies

$$(4.8) \quad 2\pi A_0^2 + \pi \sum_{n=1}^{\infty} r^{2n} (A_n^2 + B_n^2) \leq \epsilon^2.$$

Let  $\epsilon$  be very small. Let  $s = s(\epsilon)$  be the positive number (not necessarily an integer) for which



$$(4.9) \quad 1 + s^2 = r^{2s}\epsilon^{-2}.$$

If  $\epsilon^2 < 1$ , this equation uniquely determines  $s(\epsilon)$  as a function of  $\epsilon$  that increases to infinity as  $\epsilon$  decreases to zero.

In terms of  $s(\epsilon)$ , we bound  $\|u\|$  as follows.

$$(4.10) \quad \begin{aligned} \|u\|^2 &= 2\pi A_0^2 + \pi \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \\ &= 2\pi A_0^2 + \pi \sum_{n \leq s} (A_n^2 + B_n^2) + \pi \sum_{n > s} (A_n^2 + B_n^2). \end{aligned}$$

For  $n \leq s$ , we have  $r^{2n} \geq r^{2s}$ . Therefore,

$$(4.11) \quad \begin{aligned} 2\pi A_0^2 + \pi \sum_{n \leq s} (A_n^2 + B_n^2) &\leq \left[ 2\pi A_0^2 + \pi \sum_{n \leq s} r^{2n} (A_n^2 + B_n^2) \right] r^{-2s} \\ &\leq \epsilon^2 r^{-2s} \end{aligned}$$

by (4.8). Similarly, for  $n > s$  we have  $1 + n^2 > 1 + s^2$ . Therefore,

$$(4.12) \quad \begin{aligned} \pi \sum_{n > s} (A_n^2 + B_n^2) &\leq \pi \sum_{n > s} (1 + n^2) (A_n^2 + B_n^2) \cdot (1 + s^2)^{-1} \\ &\leq (1 + s^2)^{-1} \end{aligned}$$

by (4.7). Formulas (4.10)–(4.12) now yield

$$\|u\|^2 \leq \epsilon^2 r^{-2s} + (1 + s^2)^{-1}.$$

By (4.9), this becomes

$$(4.13) \quad \|u\|^2 \leq 2(1 + s^2)^{-1}.$$

It is now necessary to have the asymptotic behavior of  $s(\epsilon)$  as  $\epsilon \rightarrow 0$ . From (4.8) we find

$$2s \log(1/r) + \log(1 + s^2) = 2 \log(1/\epsilon).$$

Therefore, as  $\epsilon \rightarrow 0$ , since  $\log(1 + s^2) = o(s)$ ,

$$(4.14) \quad s(\epsilon) \sim \log \frac{1}{\epsilon} / \log \frac{1}{r}.$$

From (4.13) we now find, for sufficiently small  $\epsilon > 0$ ,  $\|u\| \leq 2^{1/2} s^{-1}(\epsilon)$  and hence

$$(4.15) \quad \|u\| < C_1 \left( \log \frac{1}{\epsilon} \right)^{-1}$$

if  $C_1$  is any constant such that

$$(4.16) \quad C_1 > 2^{1/2} \log(1/r).$$

Accordingly,

$$(4.17) \quad \rho(\epsilon) < C_1 \left( \log \frac{1}{\epsilon} \right)^{-1}$$

Next we obtain a lower bound for  $\rho(\epsilon)$ . Let  $n(\epsilon)$  be the least integer such that

$$1 + n^2 \geq r^{2n}\epsilon^{-2}.$$

Now define  $A(\epsilon)$  in terms of  $n(\epsilon)$  by

$$A = [\pi(1 + n^2)]^{-1/2}$$

and define  $\varphi(\theta) = A \cos n\theta$ . Then

$$\Omega^2(\varphi) \leq \pi A^2(1 + n^2) = 1.$$

Moreover,

$$\|K\varphi\|^2 = \pi A^2 r^{2n} = (1 + n^2)^{-1} r^{2n} \leq \epsilon^2.$$

Since  $\Omega(\varphi) \leq 1$  and  $\|K\varphi\| \leq \epsilon$ , we have

$$(4.18) \quad \rho(\epsilon) \geq \|\varphi\| = \pi^{1/2} A.$$

As  $\epsilon \rightarrow 0$ , we have

$$\pi^{1/2} A = (1 + n^2)^{-1/2} \sim n^{-1}(\epsilon).$$

But  $n(\epsilon)$  lies between  $t$  and  $t + 1$  if  $t$  is the solution of

$$1 + t^2 = r^{2t} \epsilon^{-2}.$$

The number  $t(\epsilon)$  has the asymptotic form

$$t(\epsilon) \sim \log \frac{1}{\epsilon} / \log \frac{1}{r} \text{ as } \epsilon \rightarrow 0.$$

Since  $t(\epsilon) \leq n(\epsilon) < t(\epsilon) + 1$ , we have  $n(\epsilon) \sim t(\epsilon)$ . Now (4.18) implies, for all sufficiently small  $\epsilon > 0$ ,

$$(4.19) \quad \rho(\epsilon) > C_0 \left( \log \frac{1}{\epsilon} \right)^{-1}$$

if  $C_0$  is any constant such that

$$(4.20) \quad C_0 < \log \frac{1}{r}.$$

Summarizing (4.17) and (4.19), we have, as  $\epsilon \rightarrow 0$ ,

$$(4.21) \quad C_0 \left( \log \frac{1}{\epsilon} \right)^{-1} < \rho(\epsilon) < C_1 \left( \log \frac{1}{\epsilon} \right)^{-1}.$$

A numerical example will show how this inequality determines the rate of convergence. Let  $r = 1/2$ . Assume that the unknown solution,  $u_0$ , satisfies

$$\Omega(u_0) \leq 10, \quad \|Ku_0 - g\| \leq \epsilon.$$

In this class of solutions, the worst error obtainable by Tikhonov's method is

$$(4.22) \quad \sup \|u - u_0\| = \omega\sigma(\epsilon/\omega, \alpha) = 10\sigma(\epsilon/10, \alpha).$$

Now (4.5) yields

$$(4.23) \quad 10\rho(\epsilon/10) \leq 10\sigma(\epsilon/10, \alpha) \leq \omega' \rho(\epsilon'/\omega'),$$

where

$$\omega' = (1 + \sqrt{1 + 1/\lambda})10 \text{ and } \epsilon' = (1 + \sqrt{1 + \lambda})\epsilon,$$

where  $\lambda = \alpha\omega^2/\epsilon^2 = 100\alpha/\epsilon^2$ .

By the assumption  $r = 1/2$ , we may use (4.20) to define

$$C_0 = 0.692 < \log(1/r) = \log 2.$$

Now (4.19) yields

$$\rho(\epsilon) > 0.692 \left( \log \frac{1}{\epsilon} \right)^{-1}.$$

Now (4.22) and (4.23) imply, as  $\epsilon \rightarrow 0$ ,

$$(4.24) \quad \sup \|u - u_0\| > 6.92 \left( \log \frac{10}{\epsilon} \right)^{-1}.$$

This is the lower bound for the rate of convergence; it holds for *every* value of the parameter  $\alpha$ .

The right-hand side of (4.23) gives an upper bound. If we choose  $\alpha = \epsilon^2/\omega^2 = 10^{-2}\epsilon^2$ , then (4.23) implies

$$(4.25) \quad 10\sigma(\epsilon/10, \alpha) \leq (1 + \sqrt{2})10\rho(\epsilon/10).$$

According to (4.16), if we choose

$$C_1 > 2^{1/2} \log(1/r) = 2^{1/2} \log$$

then as  $\epsilon \rightarrow 0$ ,

$$(4.26) \quad \rho(\epsilon) < C_1 \left( \log \frac{1}{\epsilon} \right)^{-1}.$$

It suffices to take  $C_1 = 1$ ; then (4.25) and (4.26) imply the upper bound

$$(4.27) \quad \sup \|u - u_0\| < 24.2 \log(10/\epsilon).$$

**5. The Backward Heat Equation.** Let  $\psi(x, t)$  satisfy the heat equation

$$(5.1) \quad \partial\psi/\partial t = \partial^2\psi/\partial x^2$$

for  $0 < x < \pi$  and for  $0 < t < T$ . Let  $\psi$  satisfy some boundary conditions, for instance:

$$(5.2) \quad \partial\psi/\partial x = 0 \text{ at } x = 0 \text{ and } x = \pi.$$

Let the initial and final temperature be denoted:

$$(5.3) \quad u_0(x) = \psi(x, 0), \quad g_0(x) = \psi(x, T).$$

We will apply Tikhonov's method to the ill-posed problem of determining  $u_0(x)$  from a function  $g(x)$  that is very near to  $g_0(x)$ :

$$(5.4) \quad \|g(x) - g_0(x)\| \leq \epsilon.$$

We suppose  $\Omega(u_0) < \infty$ , where  $\Omega$  is some functional of the form (2.3).

In this example, we are allowing  $\Omega$  to have the general form (2.3), whereas in the preceding example we assumed  $p = 1$  for simplicity. Here we will see how the order  $p$  may affect the rate of convergence in Tikhonov's method.

If  $u_0(x)$  has the Fourier series

$$(5.5) \quad u_0(x) = \sum_0^\infty A_n \cos nx$$

then  $g_0(x)$  has the Fourier series

$$(5.6) \quad g_0(x) = \sum_0^\infty A_n e^{-n^2 T} \cos nx.$$

Thus,  $u_0(x)$  is related to  $g_0(x)$  by an integral equation

$$(5.7) \quad Ku_0(x) \equiv \int_0^\pi k(x, y)u_0(y)dy = g_0(x)$$

where

$$k(x, y) = \frac{1}{\pi} \left( 1 + 2 \sum_{n=1}^\infty e^{-n^2 T} \cos nx \cos ny \right).$$

In Tikhonov's method, one chooses some positive value for the parameter  $\alpha$ , and one chooses  $u$  to minimize

$$(5.8) \quad \|Ku - g\|^2 + \alpha\Omega^2(u).$$

According to Section 3, if  $\sigma(\epsilon, \alpha)$  is the modulus of convergence, and if  $u_0$  and  $g$  satisfy

$$\Omega(u_0) \leq \omega \quad \text{and} \quad \|Ku_0 - g\| \leq \epsilon,$$

then

$$(5.9) \quad \sup \|u - u_0\| = \omega\sigma(\epsilon/\omega, \alpha).$$

As usual, the modulus of convergence,  $\sigma(\epsilon, \alpha)$ , is bounded in terms of the modulus of regularization,  $\rho(\epsilon)$ .

Let the functions  $a_i(x)$ , which are used to define  $\Omega$ , satisfy the inequalities

$$(5.10) \quad 0 < m_i \leq a_i(x) \leq M_i \quad (i = 1, \dots, p).$$

To obtain an upper bound for  $\rho(\epsilon)$ , we will use the minima,  $m_i$ . Let  $\varphi(x)$  have the Fourier series

$$\varphi(x) = \sum_{n=0}^\infty c_n \cos nx \quad (0 \leq x \leq \pi)$$

and let  $\Omega(\varphi) \leq 1$ . Then

$$(5.11) \quad \int_0^\pi [m_0\varphi^2 + \dots + m_p(\varphi^{(p)})^2] dx \leq \Omega^2(\varphi) \leq 1.$$

Let us now define the polynomial

$$(5.12) \quad F(z) = m_0 + m_1z + \dots + m_pz^p.$$

Then (5.11) gives

$$(5.13) \quad \pi m_0 c_0^2 + \frac{\pi}{2} \sum_{n=1}^\infty F(n^2) c_n^2 \leq 1.$$

If we suppose also that  $\|K\varphi\| \leq \epsilon$ , then

$$(5.14) \quad \pi c_0^2 + \frac{\pi}{2} \sum_{n=1}^\infty e^{-2n^2 T} c_n^2 \leq \epsilon^2.$$

Let  $s = s(\epsilon)$  be the positive number satisfying

$$(5.15) \quad F(s^2) = e^{-2s^2 T} \epsilon^{-2}.$$

The number  $s(\epsilon)$  is uniquely defined if  $m_0\epsilon^2 < 1$ . As  $\epsilon$  decreases to zero, the function  $s(\epsilon)$  increases to infinity. We now write

$$\begin{aligned} \|\varphi\|^2 &= \pi c_0^2 + \frac{\pi}{2} \sum_{n \leq s} c_n^2 + \frac{\pi}{2} \sum_{n > s} c_n^2 \\ &\leq \left( \pi c_0^2 + \frac{\pi}{2} \sum_{n \leq s} e^{-2n^2 T} c_n^2 \right) \cdot e^{2s^2 T} \\ &\quad + \left( \frac{\pi}{2} \sum_{n > s} F(n^2) c_n^2 \right) \cdot (F(s^2))^{-1}. \end{aligned}$$

The inequalities (5.13)–(5.15) now imply

$$(5.16) \quad \|\varphi\|^2 \leq \epsilon^2 e^{2s^2 T} + (F(s^2))^{-1} = 2(F(s^2))^{-1}.$$

We now need the asymptotic behavior of  $s(\epsilon)$ . From Eq. (5.15), we find

$$(5.17) \quad \log F(s^2) + 2s^2 T = 2 \log \frac{1}{\epsilon}.$$

Therefore, as  $\epsilon \rightarrow 0$ ,

$$s^2 T (1 + o(1)) = \log \frac{1}{\epsilon}$$

which gives the asymptotic form

$$(5.18) \quad s^2 \sim \frac{1}{T} \log \frac{1}{\epsilon} \quad \text{as } \epsilon \rightarrow 0.$$

Therefore, as  $\epsilon \rightarrow 0$ ,

$$(5.19) \quad F(s^2) \sim m_p s^{2p} \sim m_p \left( \frac{1}{T} \log \frac{1}{\epsilon} \right)^p$$

The inequality (5.16) now implies

$$(5.20) \quad \|\varphi\| \leq (1 + o(1)) 2^{1/2} m_p^{-1/2} T^{p/2} \left( \log \frac{1}{\epsilon} \right)^{-p/2},$$

where  $o(1)$  generically denotes a function of  $\epsilon$  tending to zero as  $\epsilon \rightarrow 0$ . Since this inequality holds for all  $\varphi$  satisfying  $\Omega(\varphi) \leq 1$  and  $\|K\varphi\| \leq \epsilon$ , the right-hand side of (5.20) is an upper bound for the modulus of regularization,  $\rho(\epsilon)$ .

To obtain a lower bound for  $\rho(\epsilon)$ , let

$$G(z) = M_0 + M_1 z + \cdots + M_p z^p.$$

Let  $t$  be the real number solving  $G(t^2) = e^{-2t^2 T} \epsilon^{-2}$ . The function  $t = t(\epsilon)$  increases to infinity as  $\epsilon \rightarrow 0$ . Let  $n = n(\epsilon)$  be the integer satisfying  $t(\epsilon) \leq n(\epsilon) < 1 + t(\epsilon)$ . Now define the function

$$\varphi(x) = A \cos nx = A(\epsilon) \cos n(\epsilon)x,$$

where

$$A(\epsilon) = \left( \frac{\pi}{2} G(n^2(\epsilon)) \right)^{-1/2}$$

Then

$$\Omega^2(\varphi) \leq \frac{\pi}{2} A^2 G(n^2) = 1.$$

Moreover, since  $n(\epsilon) \geq t(\epsilon)$ ,

$$\begin{aligned} \|K\varphi\|^2 &= \frac{\pi}{2} A^2 e^{-2n^2 T} = (G(n^2))^{-1} e^{-2n^2 T} \\ &\leq (G(t^2))^{-1} e^{-2t^2 T} = \epsilon^2. \end{aligned}$$

Now, since  $\Omega(\varphi) \leq 1$  and  $\|K\varphi\| \leq 1$ , we have

$$\rho(\epsilon) \geq \|\varphi\| = \left(\frac{\pi}{2} A^2\right)^{1/2}$$

Therefore, if  $n = n(\epsilon)$ ,

$$(5.21) \quad \rho(\epsilon) \geq (G(n^2))^{-1/2}.$$

To obtain the asymptotic form of  $t(\epsilon)$ , we proceed just as we did for  $s(\epsilon)$ , only replacing  $F$  by  $G$ . We then find, as  $\epsilon \rightarrow 0$ ,

$$t^2 \sim \frac{1}{T} \log \frac{1}{\epsilon}, \quad G(t^2) \sim M_p \left(\frac{1}{T} \log \frac{1}{\epsilon}\right)^p.$$

Since  $n(\epsilon) \sim t(\epsilon)$ , we have  $G(n^2) \sim G(t^2)$ . Now the inequality (5.21) yields

$$(5.22) \quad \rho(\epsilon) \geq (1 + o(1)) M_p^{-1/2} T^{p/2} \left(\log \frac{1}{\epsilon}\right)^{-p/2}.$$

This lower bound is of the same asymptotic order as the upper bound implied by (5.20):

$$(5.23) \quad \rho(\epsilon) \leq (1 + o(1)) 2^{1/2} m_p^{-1/2} T^{p/2} \left(\log \frac{1}{\epsilon}\right)^{-p/2}.$$

To estimate the rate of convergence, we again use the inequality

$$(5.24) \quad \omega\rho(\epsilon/\omega) \leq \omega\sigma(\epsilon/\omega, \alpha) \leq \omega'\rho(\epsilon'/\omega', \alpha)$$

where, if  $\lambda = \alpha\omega^2/\epsilon^2$ ,

$$\epsilon' = (1 + \sqrt{1 + \lambda})\epsilon \quad \text{and} \quad \omega' = (1 + \sqrt{1 + 1/\lambda})\omega.$$

The last few formulas show that the error tends to zero like  $(\log(1/\epsilon))^{-p/2}$ , where  $p$  is the order of the highest derivative appearing in the functional  $\Omega^2(u)$ .

**6. Arbitrarily Slow Convergence.** In view of the slow convergence of Tikhonov's method for harmonic continuation and for the backward heat equation, one may inquire just how slow the convergence may be in other applications. The answer is—arbitrarily slow.

Given any function  $\rho_0(\epsilon)$  tending monotonely to zero as  $\epsilon \rightarrow 0$ , and given the functional  $\Omega^2(u)$  of the form (2.3), we will show how to construct an integral operator  $K$  such that

$$(6.1) \quad \|K\varphi_n\| \leq \epsilon_n, \quad \Omega(\varphi_n) \leq 1,$$

and yet

$$(6.2) \quad \|\varphi_n\| \geq \rho_0(\epsilon_n)$$

for a sequence of functions  $\varphi_n(x)$  and for a sequence of numbers  $\epsilon_n$  tending to zero.

We define

$$(6.3) \quad \varphi_n(x) = A_n \sin nx \quad (0 < x < \pi)$$

where  $A_n$  will be chosen to make

$$(6.4) \quad \Omega(\varphi_n) \leq 1 \quad \text{and} \quad \|\varphi_n\| \geq \rho_0(\epsilon_n).$$

If  $a_0(x) \geq m_0 > 0$ , then

$$\Omega^2(u) \geq m_0 \|u\|^2.$$

Therefore,  $\Omega^2(A_n \sin nx) \geq m_0 A_n^2 \pi/2$  while  $\|A_n \varphi_n\|^2 = A_n^2 \pi/2$ . To achieve (6.4), we require that  $A_n$  satisfy

$$(6.5) \quad \left(m_0 \frac{\pi}{2}\right)^{1/2} A_n \leq 1 \quad \text{and} \quad \left(\frac{\pi}{2}\right)^{1/2} A_n \geq \rho_0(\epsilon_n).$$

This can be achieved if and only if  $\rho_0(\epsilon_n)$  satisfies

$$(6.6) \quad \rho_0(\epsilon_n) \leq m_0^{-1/2}.$$

Let  $\{\epsilon_n\}$  be any sequence of positive numbers tending monotonely to zero such that (6.6) is satisfied for  $n = 1$  (and hence for  $n > 1$ ). Now choose

$$A_n = \frac{1}{2} \left[ \left(\frac{\pi}{2}\right)^{-1/2} \rho_0(\epsilon_n) + \left(m_0 \frac{\pi}{2}\right)^{-1/2} \right].$$

Then (6.5) and hence (6.4) are satisfied.

We now define the kernel

$$(6.7) \quad k(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \lambda_n \sin nx \sin ny$$

where  $\lambda_n > 0$  and  $\sum \lambda_n < \infty$ . We will now choose the numbers  $\lambda_n$  to make

$$\|K\varphi_n\| = \left\| \int_0^\pi k(x, y) \varphi_n(y) dy \right\| \leq \epsilon_n.$$

We have

$$K\varphi_n = K(A_n \sin nx) = \lambda_n A_n \sin nx,$$

$$\|K\varphi_n\| = \lambda_n (\pi/2)^{1/2} A_n.$$

Therefore,  $\|K\varphi_n\| \leq \epsilon_n$  if the  $\lambda_n$  are chosen to satisfy  $\lambda_n \leq (\pi/2)^{-1/2} A_n^{-1} \epsilon_n$ . This completes the construction.

If Tikhonov's method, using the functional  $\Omega$ , is applied to the integral equation  $Ku_0 = g_0$ , we know from Section 3 that, for every choice of the parameter  $\alpha$ , the modulus of convergence is bounded below by the modulus of regularization:

$$\sigma(\epsilon, \alpha) \geq \rho(\epsilon).$$

Now the inequalities (6.1) and (6.2) imply

$$(6.8) \quad \sigma(\epsilon_n, \alpha) \geq \rho(\epsilon_n) \geq \rho_0(\epsilon_n) \quad (n = 1, 2, \dots).$$

Thus, convergence in Tikhonov's method can be arbitrarily slow. In the next section we will consider an application in which the convergence is fairly fast.

7. **Numerical Differentiation.** Jane Cullum [3] has considered the application of Tikhonov's method to numerical differentiation. She supposes that

$$(7.1) \quad Ku_0(x) = \int_0^x u_0(y)dy = g_0(x) \quad (0 \leq x \leq 1).$$

She also supposes that  $g(1) = 0$ . She then states the problem of minimizing

$$(7.2) \quad \|Ku_1 - g\|^2 + \left(\int_0^1 u_1 dx\right)^2 + \alpha\Omega^2(u_1)$$

where

$$(7.3) \quad \Omega^2(\varphi) = \int_0^1 (\varphi^2 + \varphi'^2)dx = \|\varphi\|^2 + \|\varphi'\|^2.$$

Assuming that the unknown function  $u_0(x)$  has a bounded derivative, she proves that there exists a constant  $Q$  such that

$$(7.4) \quad \|u_1 - u_0\| \leq Q\alpha^{1/4}.$$

Because of the inclusion of the term  $(\int u_1 dx)^2$  in (7.2), this minimization problem is different from that of finding  $u$  minimizing

$$(7.5) \quad \|Ku - g\|^2 + \alpha\Omega^2(u).$$

Let us here consider the problem of minimizing (7.5), where we will require

$$(7.6) \quad \int_0^1 u dx = 0 \quad \text{and} \quad \Omega(u) < \infty$$

and let us here suppose

$$(7.7) \quad \|g - Ku_0\| = \|g - g_0\| \leq \epsilon.$$

We wish to estimate the modulus of convergence

$$(7.8) \quad \sigma(\epsilon, \alpha) = \sup_{\Omega(u_0) \leq 1; \|g - g_0\| \leq \epsilon} \|u - u_0\|.$$

If, instead of  $\Omega(u_0) \leq 1$ , we assume  $\Omega(u_0) \leq \omega$ , then (7.8) implies

$$(7.9) \quad \omega\sigma(\epsilon/\omega, \alpha) = \sup \|u - u_0\|.$$

Here the modulus of regularization is easy to find. Assuming

$$\Omega(\varphi) \leq 1, \quad \|K\varphi\| \leq \epsilon, \quad \text{and} \quad \int_0^1 \varphi dx = 0,$$

we find

$$(7.10) \quad \|\varphi\|^2 = \int_0^1 \varphi^2 dx = - \int_0^1 \varphi'(K\varphi) dx$$

since  $K\varphi(x) = 0$  at  $x = 1$ . Therefore,

$$(7.11) \quad \|\varphi\|^2 \leq \|\varphi'\| \|K\varphi\| \leq \Omega(\varphi) \|K\varphi\| \leq \epsilon.$$

Therefore, the modulus of regularization satisfies

$$(7.12) \quad \rho^2(\epsilon) \leq \epsilon.$$

It would be easy to obtain a lower bound for  $\rho(\epsilon)$  using test-functions of the form  $\varphi = A \cos n\pi x$ . But we do not need a lower bound for  $\rho$  if we only wish to obtain an upper bound for  $\sigma$ . We now use (3.16) and (7.12) to obtain



$$\omega\sigma(\epsilon/\omega, \alpha) \leq \omega' \rho(\epsilon'/\omega') \leq (\omega' \epsilon')^{1/2}$$

where, if  $\lambda = \alpha\omega^2/\epsilon^2$ ,

$$\omega' = \omega\sqrt{1+1/\lambda} \quad \text{and} \quad \epsilon' = \epsilon\sqrt{1+\lambda}.$$

Therefore,

$$(7.13) \quad \omega\sigma(\epsilon/\omega, \alpha) \leq [(1+1/\lambda)(1+\lambda)\omega\epsilon]^{1/2}.$$

If  $\lambda = 1$ , i.e. if  $\alpha = (\epsilon/\omega)^2$ , the right-hand side of (7.13) takes its least value. Then (7.13) and (7.9) imply

$$(7.14) \quad \sup_{\Omega(u_0)} \|u - u_0\| \leq 2(\omega\epsilon)^{1/2} = 2\omega\alpha^{1/4}.$$

Here we obtain an upper bound of the order  $\alpha^{1/4}$ , as does Cullum.

Under stronger assumptions, we can obtain a stronger inequality by the elementary method of centered differences. Let us suppose again  $u_0(x) = g'_0(x)$  and suppose that we are given data  $g(x)$  such that

$$(7.15) \quad |g(x) - g_0(x)| \leq \epsilon \quad (a < x < b).$$

Moreover, suppose we have a bound for the second derivative:

$$|u''_0(x)| \leq M \quad (a < x < b).$$

(This makes the comparison to Cullum's result unfair, since she makes no assumption that  $u_0(x)$  has a second derivative.) We then use the estimate

$$(7.16) \quad u(x) = (2h)^{-1}(g(x+h) - g(x-h))$$

with the increment

$$(7.17) \quad h = (3\epsilon/M)^{1/3}.$$

We now assert

$$(7.18) \quad |u(x) - u_0(x)| \leq \frac{1}{2} 3^{2/3} M^{1/3} \epsilon^{2/3}.$$

This is better than the order  $\epsilon^{1/2} = \alpha^{1/4}$  appearing in (7.14).

To prove the assertion (7.18), let  $g(x) = g_0(x) + f(x)$ . Then

$$\begin{aligned} u(x) - u_0(x) &= (2h)^{-1} [g_0(x+h) - g_0(x-h)] - u_0(x) \\ &\quad + (2h)^{-1} [f(x+h) - f(x-h)]. \end{aligned}$$

By (7.15), we have  $|f| = |g - g_0| \leq \epsilon$ . And since  $u_0 = g'_0$ , Taylor's theorem implies

$$(2h)^{-1} [g_0(x+h) - g_0(x-h)] = u_0(x) + \frac{1}{6} h^2 u''_0(x + \theta h)$$

where  $-1 < \theta \leq 1$ . Therefore, since  $|u''_0| \leq M$ ,

$$|u(x) - u_0(x)| \leq \frac{1}{6} h^2 M + h^{-1} \epsilon.$$

As a function of  $h$ , this is minimized by the value given in (7.17); and the minimum value given in (7.18) results.

**8. Summary and Remarks.** We have shown that convergence in Tikhonov's method is determined by the modulus of regularization, which measures the degree to which boundedness of the functional  $\Omega(\varphi)$  permits the inversion of the operator  $K$  applied to  $\varphi$ . For harmonic continuation and for the backward heat equation, we have found that the error of approximation tends to zero like a power of  $1/(\log(1/\epsilon))$  if  $\epsilon$  is the norm of the data-error (or equivalent rounding error). For numerical differentiation, we have obtained a comparison of Tikhonov's method with the elementary method of centered differences.

Tikhonov's method states the problem of minimizing

$$(8.1) \quad \|Ku - g\|^2 + \alpha\Omega^2(u).$$

The discussion in Section 3 shows that there is not necessarily any advantage in *minimizing* this functional if, by any easier means, one can obtain a function  $u_1$  satisfying inequalities

$$(8.2) \quad \|Ku_1 - g\| \leq \epsilon_1,$$

$$(8.3) \quad \Omega(u_1) \leq \omega_1,$$

where  $\epsilon_1$  is small and where  $\omega_1$  is not too large. For then, if

$$(8.4) \quad \|Ku_0 - g\| \leq \epsilon \text{ and } \Omega(u_0) \leq \omega$$

we can deduce  $\|K(u_1 - u_0)\| \leq \epsilon_1 + \epsilon$  and  $\Omega(u_1 - u_0) \leq \omega_1 + \omega$  and consequently

$$\mu(u_1 - u_0) \leq (\omega + \omega_1)\rho((\epsilon + \epsilon_1)/(\omega + \omega_1))$$

where  $\mu(u_1 - u_0)$  is some norm of the error, and where  $\rho$  is the related modulus of regularization.

The Tikhonov functional (8.1) is, of course, the *Lagrange* functional for the minimization of  $\Omega^2(u)$  if a value is prescribed for  $\|Ku - g\|^2$ . Indeed, if  $u = u_1$  minimizes the functional (8.1), and if

$$\|Ku_1 - g\| = \epsilon_1 \text{ and } \Omega(u_1) = \omega_1$$

then the only function  $u_2$  satisfying both inequalities

$$(8.5) \quad \|Ku_2 - g\| \leq \epsilon_1 \text{ and } \Omega(u_2) \leq \omega_1$$

is  $u_2 = u_1$  because the inequalities (8.5) imply that  $u_2$  also minimizes the functional (8.1), and now the uniqueness of the minimizing solution (proved in Section 2) implies  $u_2 = u_1$ .

The applications in Sections 4 and 5 show that one must not necessarily expect rapid convergence from Tikhonov's method. In Section 6 it is proved that arbitrarily slow convergence is possible.

If one knows only that the unknown solution,  $u_0$ , satisfies (8.4), a good choice for the parameter  $\alpha$  is

$$\alpha = (\epsilon/\omega)^2.$$

Then the upper and lower bounds (3.17) for the modulus of convergence are nearly equal; the lower bound is independent of  $\alpha$ .

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