

## A Nonexistence Theorem for Explicit $A$ -Stable Methods

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**Abstract.** It is proved that there are no  $A$ -stable explicit methods in a general class of "linear" methods. The class contains, for example, Runge-Kutta methods, linear multistep methods, predictor-corrector formulas, cyclic multistep methods and linear multistep methods with higher derivatives.

It is obvious that explicit methods for stiff problems of ordinary differential equations have to be nonlinear in some sense. In this paper, we prove that there are no  $A$ -stable explicit methods in a general class of "linear" methods. Consider the following methods

$$(1) \quad \sum_{r=0}^s \sum_{i=0}^k h^r A_i^r y_{n-k+i}^{(r)} = 0$$

for the solution of  $x' = f(t, x)$ ,  $x(0) = x_0$ ,  $t \in [0, 1]$ . In Eq. (1),  $y_n^{(0)}$  is an approximation to

$$x(t_n) = \begin{pmatrix} x(t_n^1) \\ \vdots \\ x(t_n^m) \end{pmatrix}$$

and  $y_n^{(r)}$ ,  $r = 1(1)s$ , a corresponding approximation to  $x^{(r)}(t_n)$ , the  $t_n^j = (n-1 + \tau_j)h$ ,  $\tau_j$  fixed constants. The  $A_i^r = (a_{\mu\lambda}^{ir})$  are fixed matrixes with

$$a_{\mu\lambda}^{k0} = 0 \quad \text{for } \mu < \lambda, \quad a_{\mu\mu}^{k0} \neq 0, \quad \mu = 1(1)m,$$

$$\left( \sum_{i=0}^k A_i^0 \right) e = 0, \quad \text{where } e = (1, 1, \dots, 1)^T.$$

The method is explicit if

$$a_{\mu\lambda}^{kr} = 0 \quad \text{for } \mu \leq \lambda, \quad r = 1(1)s.$$

If  $s = 1$ , method (1) reduces to Stetter's simple  $m$ -stage  $k$ -step method, cf. [2, p. 275] It thus contains Runge-Kutta methods, linear multistep methods, predictor-corrector

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formulas and cyclic multistep methods. In addition, method (1) contains, e.g., multistep methods with higher derivatives. We shall prove

**THEOREM.** *There are no explicit A-stable methods (1).*

*Remark 1.* For linear multistep methods, the result has been proved by Dahlquist [1].

*Remark 2.* The predictor-corrector methods, when the corrector is iterated  $m$ -times, are explicit in formulation (1). Thus, they are not  $A$ -stable, although the corrector formula itself might be  $A$ -stable.

*Remark 3.* In fact, we shall prove that there are no explicit  $A(0)$ -stable [3] methods (1).

*Proof.* Let us consider the scalar equation  $x' = \lambda x$ ,  $\text{Re } \lambda < 0$ . Applying method (1) to this equation we get the linear difference equation of order  $k$

$$(2) \quad \sum_{r=0}^s \sum_{i=0}^k (\lambda h)^r A_i^r y_{r-k+i} = 0.$$

We can assume that  $\sum_{r=0}^s (\lambda h)^r A_k^r$  is nonsingular. Thus, Eq. (2) can be expressed in the form

$$\begin{aligned} \begin{pmatrix} y_{n-k+1} \\ \vdots \\ y_n \end{pmatrix} &= \begin{pmatrix} 0 & & I & & 0 \\ \vdots & \cdot & \cdot & \cdot & \cdot \\ \vdots & & & & \\ \vdots & & & & \\ 0 & & & 0 & I \\ -B_k^{-1}B_0 & \cdots & -B_k^{-1}B_{k-1} & & \end{pmatrix} \begin{pmatrix} y_{n-k} \\ \vdots \\ y_{n-1} \end{pmatrix} \\ &= C(\lambda h) \begin{pmatrix} y_{n-k} \\ \vdots \\ y_{n-1} \end{pmatrix}, \end{aligned}$$

where  $B_i(\lambda h) = \sum_{r=0}^s (\lambda h)^r A_i^r$ ,  $i = 0(1)k$ .  $A$ -stability requires  $\|C(\lambda h)\| < 1$ , which demands that all eigenvalues of  $C(\lambda h)$  have modulus smaller than 1. Hence, we consider these eigenvalues, which are the roots of

$$(3) \quad \det \left[ \sum_{r=0}^s \sum_{i=0}^k (\lambda h)^r A_i^r z^i \right] = 0,$$

since

$$\det [zI - C(\lambda h)] = \det \left( \sum_{i=0}^k B_k^{-1} B_i z^i \right) = \det(B_k^{-1}) \cdot \det \left( \sum_{i=0}^k B_i z^i \right).$$

Next, we show that for explicit methods (1) Eq. (3) is of the form

$$(4) \quad z^{km} + \sum_{\nu=0}^{k-1} p_{\nu}(\lambda h) z^{\nu} = 0,$$

where the  $p_{\nu}$ 's are polynomials. Remember that for an explicit method  $a_{\mu\lambda}^{kr} = 0$  for  $\mu \leq \lambda$ ,  $r = 1(1)s$  and  $a_{\mu\lambda}^{k0} = 0$  for  $\mu < \lambda$ . Let us write Eq. (3) as

$$\det \begin{pmatrix} a_{11}^{k0} z^k + q_{11}^{k-1} & q_{12}^{k-1} & \dots & q_{1m}^{k-1} \\ q_{21}^k & a_{22}^{k0} z^k + q_{22}^{k-1} & \dots & q_{2m}^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1}^k & q_{m2}^k & \dots & a_{mm}^{k0} z^k + q_{mm}^{k-1} \end{pmatrix} = 0,$$

where  $q_{\mu\omega}^j = \sum_{r=0}^s \sum_{i=0}^j (\lambda h)^r a_{\mu\omega}^{ir} z^i$ . Expanding according to the elements of the first column, we observe that the only cofactor containing  $z^k$  in the elements of the first row is that corresponding to the first element. Denote this cofactor by  $C_{11}$ . Thus, the only term in the expansion that can contain  $z^{km}$  is  $a_{11} z^k C_{11}$ . The same reasoning applies directly to  $C_{11}$  and so on, giving the coefficient of  $z^{km}$  as  $\prod_{\mu=1}^m a_{\mu\mu}^{k0}$ . Dividing the expansion by  $\prod_{\mu=1}^m a_{\mu\mu}^{k0}$ , which is different from zero by assumption, gives Eq. (3) in form (4).

Suppose not all  $p_{\nu}$ 's are constant in representation (4) of Eq. (3). Then at least one of the roots of Eq. (3) grows without bound in modulus, for example when  $\lambda h \rightarrow -\infty$  in the real axis. On the other hand, if none of the  $p_{\nu}$ 's depend on  $\lambda h$ , then Eq. (3) does not depend on  $\lambda h$  and is thus equivalent to

$$(5) \quad \det \left[ \sum_{i=0}^k A_i^0 z^i \right] = 0,$$

which is obtained from (3) by substitution of  $\lambda h = 0$ . Equation (5) is clearly satisfied by  $z = 1$  since, by assumption,  $(\sum_{i=0}^k A_i^0) e = 0$ , where  $e = (1, 1, \dots, 1)^j$ . This is again in contradiction with the requirement of A-stability.

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1. G. DAHLQUIST, "A special stability problem for linear multistep methods," *Nordisk Tidsskr. Informationsbehandling (BIT)*, v. 3, 1963, pp. 27-43. MR 30 #715.
2. H. J. STETTER, *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer, Berlin, 1973.
3. O. B. WIDLUND, "A note on unconditionally stable linear multistep methods," *Nordisk Tidsskr. Informationsbehandling (BIT)*, v. 7, 1967, pp. 65-70. MR 35 #6373.