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The Design of Slowly Shrinking Labelled Squares*

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Abstract. This paper considers the nonlinear operator T, which acts on 4-dimensional vectors with nonnegative real components. T transforms such a vector into a new vector whose components are the magnitudes of the differences of cyclically adjacent pairs of components of the previous vector. We show that T is not nilpotent.

This short paper is devoted to a problem which I first heard from the Lehmers, Professor D. Gale, and Professor A. P. Morse on May 6, 1973. It consumed the greater part of that evening and it was not solved until the next day.

If the four vertices of a square are labelled with a given set of four nonnegative real numbers, then one may label the midpoint of each side as the absolute value of the difference between the labels on the corresponding vertices. Taking these labels on the midpoints of the original square as the labels on the vertices of a new square, the operation may be repeated again and again. For example, if the vertices of the original square are labelled, in clockwise order, as $\mathbf{x} = [1, 5, 10, 8]$, then the labels after successive shrinkings are

$$\begin{aligned} \mathbf{x}T &= [4, 5, 2, 7], \\ \mathbf{x}T^2 &= [1, 3, 5, 3], \\ \mathbf{x}T^3 &= [2, 2, 2, 2], \\ \mathbf{x}T^4 &= [0, 0, 0, 0]. \end{aligned}$$

If one knows the ordering of the original labels, then the shrinking operation can be represented as a linear transformation. For example, if $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$, then

$$\mathbf{x}T = \mathbf{x} \begin{bmatrix} -1 & 0 & 0 & -1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & +1 \end{bmatrix}.$$

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On the other hand, if $x_1 \geqslant x_2 \geqslant x_4 \geqslant x_3$, then

$$\mathbf{x}T = \mathbf{x}\begin{bmatrix} +1 & 0 & 0 & +1 \\ -1 & +1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{bmatrix}.$$

Almost any square labelled with positive integers will shrink to zero after only a few applications of the T operator, as the reader is invited to verify. One might even be tempted to conjecture that the nonlinear operator T is nilpotent, i.e., some finite power of it will shrink any square to zero. This conjecture turns out to be false, but the following weakened form of it is true:

LEMMA. *If the components of* $\mathbf{x}$ *with maximum and minimum labels are not cyclically adjacent, then* $\mathbf{x}T^6 = \mathbf{0}$.

Proof. If the extreme labels are not adjacent they must be opposite, and without loss of generality we may assume that $x_1 \geqslant x_2 \geqslant x_4 \geqslant x_3$. Setting $y_i = x_i - x_3$, we have

$$y_1 \geqslant y_2 \geqslant y_4 \geqslant y_3 = 0,$$

$$\mathbf{x}T = \mathbf{y}T = [y_1 - y_2, y_2, y_4, y_1 - y_4],$$

$$\mathbf{x}T^2 = [|y_1 - 2y_2|, y_2 - y_4, |y_1 - 2y_4|, y_2 - y_4].$$

If we then set $z_1 = ||y_1 - 2y_2| + y_4 - y_2|$ and $z_3 = ||y_1 - 2y_4| + y_4 - y_2|$ we have

$$\mathbf{x}T^3 = [z_1, z_1, z_3, z_3].$$

Setting $w = |z_1 - z_3|$ yields

$$\mathbf{x}T^4 = [0, w, 0, w], \qquad \mathbf{x}T^5 = [w, w, w, w], \qquad \mathbf{x}T^6 = 0. \qquad \text{Q.E.D.}$$

COROLLARY. *If* $\mathbf{x}T^7 \neq \mathbf{0}$, *then the components of* $\mathbf{x}$ *are cyclically ordered.*

Proof. By appropriate choice of coordinates, we may assume that the minimum label occurs at component x_1. If $\mathbf{x}T^6 \neq \mathbf{0}$, then the lemma asserts that the maximum label is not x_3, so by appropriate choice of direction we may assume that x_4 is the maximum label. So there is no loss of generality in assuming that $x_1 \leqslant x_2 \leqslant x_4$ and $x_1 \leqslant x_3 \leqslant x_4$. If $x_3 \leqslant x_2$, then

$$\mathbf{x}T = [x_2 - x_1, x_2 - x_3, x_4 - x_3, x_4 - x_1].$$

The maximum component of $\mathbf{x}T$ is $x_4 - x_1$, and the minimum component of $\mathbf{x}T$ is $x_2 - x_3$, whence by the Lemma, $\mathbf{x}T \cdot T^6 = \mathbf{0}$, which contradicts the hypothesis of this Corollary. The assumption that $x_3 \leqslant x_2$ is therefore untenable, and we can therefore conclude that $x_1 \leqslant x_2 < x_3 \leqslant x_4$. Q.E.D.

If we wish to find labellings which will not shrink to zero in n or fewer operations, then the Corollary tells us that for any $m \leqslant n - 7$, the components of $\mathbf{y} = \mathbf{x}T^m$ must be cyclically ordered. Furthermore, for $1 \leqslant m \leqslant n - 7$, the biggest component of $\mathbf{x}T^m$ must be equal to the sum of the other three components. So, without loss of generality, we may assume that $y_1 \leqslant y_2 < y_3 \leqslant y_4 = y_1 + y_2 + y_3$, and that $\mathbf{y}T = \mathbf{z} = [z_1, z_2, z_3, z_4] = \mathbf{y}M$, where M is this 4×4 matrix

$$M = \begin{bmatrix} -1 & 0 & 0 & -1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & +1 \end{bmatrix}.$$

The biggest component of $\mathbf{z}$ is obviously z_4, and if $\mathbf{z}T^6 \neq 0$, then the components of $\mathbf{z}$ must also be cyclically ordered. So either $z_1 \leqslant z_2 \leqslant z_3 \leqslant z_4$ or $z_3 \leqslant z_2 \leqslant z_1 \leqslant z_4$. To see that the latter ordering is not possible, we notice that since $y_2 = y_1 + z_1$ and $y_3 = y_1 + z_1 + z_2$ and $y_4 = y_1 + z_1 + z_2 + z_3$, the equation $y_4 = y_1 + y_2 + y_3$ implies $z_3 = 2y_1 + z_1$, whence $z_1 \leqslant z_3$. Thus, the components of $\mathbf{z}$ are cyclically ordered in the same direction as the components of y. It follows that for $n \geqslant 9$, $\mathbf{x}T^n = (\mathbf{x}T)M^{n-8}T^7$.

The eigenvalues of M are the roots of the polynomial

$$|M - \lambda I| = (\lambda + 1)^3(\lambda - 1) + 1 = 0 = \lambda(\lambda^3 + 2\lambda^2 - 2)$$

and the corresponding eigenvectors are

$$\mathbf{x} = x_1[1, (\lambda + 1), (\lambda + 1)^2, (\lambda + 1)^3].$$

One of the eigenvalues of M is 0, corresponding to the trivial eigenvector $\mathbf{x} = 1, 1, 1, 1$. There is also one positive real eigenvalue, $= .83928675521 \cdots$ and a pair of complex eigenvalues of greater magnitude. It is possible to keep $\mathbf{x}T^n \neq \mathbf{0}$ for large n only by choosing an initial vector $\mathbf{x}$ for which $\mathbf{x}T$ is very close to $\mathbf{y}$, the nontrivial real eigenvector of M. This may be accomplished either by taking $\mathbf{x}$ to be a linear combination of $\mathbf{y}$ and $[1, 1, 1, 1]$, or by taking $\mathbf{x}$ to be a linear combination of $[1, 1, 1, 1]$ and $[y_4, y_3 + y_2, y_3, 0]$. In the latter case the first application of T reverses the ordering of the labels. □

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