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On Detecting a Periodic Event by Means of Periodic Observations. I

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Abstract. The following situation is considered. A certain event is observable only intermittently according to some period. An observer attempts to detect an occurrence of the event by making observations according to some other period. We wish to determine the probability $P(s)$ that the event is first detected on the sth observation. The period of the event may be known or unknown. This paper discusses the number-theoretic calculations necessary to solve the problem; computational aspects will be discussed in a future paper.

1. Introduction. In this paper we consider the following problem. An observer attempts to detect an intermittently observable event by making periodic instantaneous observations at a fixed rate of one every p units of time. The event becomes observable every T units, remaining so for t units. The ratio t/T is a known constant $\tau < \frac{1}{2}$. The observer begins his observations at a randomly selected time. We wish to calculate for each integer s the probability $P(s)$ that the event is first detected on the sth observation. (It will be convenient to number the observations starting with zero.) We will consider first the case in which T is known; we will then turn to the case in which T is unknown, but has a known continuous probability distribution $Q(T)$. In this paper we concentrate on the number-theoretic aspects of this problem. In [1], computational aspects of the problem will be considered.

2. The Case of Constant T.

THEOREM 1. *Let $\theta = p/T$. Let n be the least integer for which $|n\theta - k| \leqslant \tau$ for some k. If $n\theta - k \neq 0$, let m be the least positive integer for which $|m\theta - l| \leqslant \tau$ for some l and for which $n\theta - k$ and $m\theta - l$ have opposite signs. Set $\tau_1 = |n\theta - k|$ and $\tau_2 = (1 - n\tau - (m - n)\tau_1)/n$. Then*

$$P(s) = \begin{cases} \tau, & 0 \leqslant s \leqslant n-1, \\ \tau_1, & n \leqslant s \leqslant m-1, \\ \tau_2, & m \leqslant s \leqslant m+n-1, \\ 0, & s \geqslant m+n. \end{cases}$$

If $n\theta - k = 0$, $P(s) = \tau$ for $0 \leqslant s \leqslant n-1$, $P(s) = 0$ for $s \geqslant n$.

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Proof. It will be convenient to transform the problem slightly by letting the observations have duration t and letting the moments of observability be instantaneous. This is clearly equivalent to the original problem.

Let the zeroth observation begin at time zero, and let the event be visible at times $\phi, \phi + T, \phi + 2T, \cdots$, where $0 \leqslant \phi < T$. By assumption, all values of ϕ are equally likely. The event will be observed at the sth observation if $sp \leqslant \phi + kt \leqslant sp + t$ for some k, or equivalently, if $s \leqslant \phi/T + k \leqslant s + \tau$.

Thus, the sth observation, if unsuccessful, eliminates a range of length t of possible values for ϕ. It will be convenient to identify times differing by an integral multiple of T; thus we can consider them as being on a circle with a time x represented by an angle $2\pi x/T$. Observation zero always sweeps out a fraction $\tau = t/T$ of the circle; typically, the next observation sweeps out a disjoint interval, and so on. The probability of first observing the event is τ for each of these observations.

Eventually, however, some interval must overlap another. Intervals u and v will overlap if $|u\theta - v\theta - z| \leqslant \tau$ for some integer z; this will first occur for $u = n$, $v = 0$, $z = k$. We will refer to this first overlap as the primary overlap.

Once overlap has occurred at n, it must occur at $n + 1, n + 2, \cdots$, since $|(n + 1)\theta - \theta - k| = |(n + 2)\theta - 2\theta - k| = \cdots = \tau_1$. Thus, the n observations $0, \cdots, n - 1$ produce n disjoint intervals, but no later observation can increase the number of disjoint intervals. Hence, in general we expect later observations to fill in the n gaps between the intervals, eventually sweeping out all possible values of ϕ. To study the manner in which this happens, we distinguish three cases: $n\theta - k = 0$, $n\theta - k > 0$, and $n\theta - k < 0$. The first case is very simple; interval n exactly coincides with interval 0, and this case of the theorem follows immediately.

We now turn to the second case: $0 < n\theta - k = \tau_1 \leqslant \tau$. The interval corresponding to n is $[n\theta - k, n\theta - k + \tau]$; that corresponding to 0 is $[0, \tau]$. Thus the primary overlap is in the forward direction; that is, the new interval overlaps the right-hand endpoint of the old. The interval at 0 has been increased in length from τ to $\tau + \tau_1$. The amount of overlap of interval $n + 1$ with interval 1 is also τ_1, and so on. Thus each interval $n, n + 1, \cdots$ will sweep out a new amount τ_1 until such an interval simultaneously overlaps the left-hand endpoint of another interval, which must be interval zero. We will call the first occurrence of this case the secondary overlap; it occurs at interval m.

Since the interval m simultaneously overlaps the right- and left-hand endpoints, respectively, of two intervals, it closes one of the n gaps between intervals. Clearly, the interval $m + 1$ closes another gap, and so on until all n gaps are closed. This occurs with interval $n + m - 1$; at this point the whole range of ϕ has been tested. We must calculate σ the length of each gap when closed. The first n observations sweep out a portion τ of the circle, the next $m - n$ sweep out τ_1, and the last n sweep out σ. Since all these must sum to one, we have $n\tau + (m - n)\tau_1 + n\sigma = 1$; so

$$\sigma = (1 - n\tau - (m - n)\tau_1)/n = \tau_2.$$

From these facts, the theorem follows in the case $0 < n\theta - k$.

In the third case, namely $n\theta - k = -\tau_1 < 0$, the analysis is essentially the same, with the words "right-hand" and "left-hand" exchanged. The condition that interval m first closes a gap, and thus determines the secondary overlap, is $0 \leqslant m\theta - k \leqslant \tau$. The formula for τ_2 is unchanged. This completes the proof.

We remark on the method of calculating the above values. For values of τ which are not too small, it is not hard to calculate n, since we may search on either n or k and in either case need try at most $[1/\tau]$ values. We may also use continued fractions, although no gain in efficiency is likely unless τ is rather small. On the other hand, m can be arbitrarily large. We therefore seek an efficient method of calculating m. Fortunately, such a method exists, which also leads to a simpler formula for τ_2. For the following theorem, we keep all the previous notation.

THEOREM 2. *Let $\tau_1 > 0$ and let i and j, $1 \leqslant j < n$, be chosen so that $|j\theta - i|$ is minimized. (If $n = 1$, take $j = 0$.) Set*

$$(1 - n\tau - j\tau_1)/n\tau_1 = K - \gamma,$$

where K is an integer and $0 \leqslant \gamma < 1$. Then $m = Kn + j$ and $\tau_2 = \tau_1(1 - \gamma)$.

Proof. We will consider only the case in which $0 < n\theta - k \leqslant \tau$; the other case is treated identically. We also ignore the case $n = 1$, which is easily treated. It is easily seen that j is unique and that $j\theta - i < 0$, so that $j\theta - i < -\tau$. We now show that there exists an $L \geqslant 1$ such that $m = Ln + j$, $l = Lk + i$. Recall that by definition m is the least positive integer such that $-\tau \leqslant m\theta - l < 0$. We have $(Ln + j)\theta - (Lk + i) = (j\theta - i) + L\tau_1$, so certainly some values of L give $-\tau \leqslant (Ln + j)\theta - (Lk + i) < 0$, since $\tau_1 \leqslant \tau$.

We must show that there exist no $x < Ln + j$ and y not of the given form such that $-\tau \leqslant x\theta - y < 0$. Indeed, we will show more: that if $L \geqslant 1$ and $(j\theta - 1) + L\tau_1 < 0$, then for no $x < Ln + j$ and y can $(j\theta - i) + L\tau_1 \leqslant x\theta - y < 0$. Assume the contrary, and let L be the smallest value for which x and y as above exist. Certainly $L > 1$, since otherwise $j\theta - i \leqslant x\theta - y - \tau_1 = (x - n)\theta - (y - k) < 0$; but obviously $n < x < n + j$, so the inequality contradicts the minimality of $|j\theta - i|$. But on the other hand, if $L > 1$, $Ln + j < x$, we deduce from $(j\theta - i) + L\tau_1 \leqslant x\theta - y < 0$ that $(j - i) + (L - 1)\tau_1 \leqslant (x - n)\theta - (y - k) < 0$, contradicting the minimality of L.

Thus we have that the m and l sought have the values $Ln + j$ and $Lk + i$, respectively, for some $L > 0$. We now must show that $K = L$. Consider the relation $n\tau + (m - n)\tau_1 + n\tau_2 = 1$. The definition of τ_2 implies that $0 < \tau_2 \leqslant \tau_1$. Hence

$$n\tau + m\tau_1 - n\tau_1 < 1 \leqslant n\tau + m\tau_1,$$

so

$$1 - n\tau \leqslant m\tau_1 < 1 - n\tau + n\tau_1;$$

thus

$$(1 - n\tau)/\tau_1 \leqslant m < (1 - n\tau)/\tau_1 + n.$$

But $m = Ln + j$, so

$$(1 - n\tau)/\tau_1 - j \leqslant Ln < (1 - n\tau)/\tau_1 + n - j;$$

thus

$$(1 - n\tau - j\tau_1)/n\tau_1 \leqslant L < (1 - n\tau - j\tau_1)/n\tau_1 + 1,$$

so that $K = L$.

Furthermore,

$$\tau_2 = \frac{1 - n\tau - (m - n)\tau_1}{n} = \tau_1 - \frac{m\tau_1 + n\tau - 1}{n}$$

$$= \tau_1 - \frac{Kn\tau_1 + j\tau_1 + n\tau - 1}{n} = \tau_1 - \tau_1 K + \frac{1 - n\tau - j\tau_1}{n} \cdot \frac{\tau_1}{\tau_1}$$

$$= \tau_1(1 - \gamma).$$

This completes the proof.

Any method of finding n and k can be made to yield j and i simultaneously with little additional effort; thus we have an efficient scheme for generating all the values pertaining to the function P.

3. The Case of Unknown T. We now consider the case in which T, and hence θ, is unknown. Designate $P(s)$, when θ is fixed, by $P_\theta(s)$. Obviously we wish to integrate each value $P_\theta(s)$ with respect to $Q(t)$. In what follows, we will consider $R(\theta) = 1 - Q(p/\theta)$ instead of $Q(t)$. However, any integration must be done carefully; as we will see, each $P_\theta(s)$ is continuous as a function of θ, but piecewise linear, having on the order of $1/\tau^2$ pieces. This sort of function is very difficult to integrate numerically with any accuracy because of the discontinuities in the derivative.

We will show that it is possible to locate the singularities of $P_\theta(t)$ as a function of θ in a fairly efficient manner and thus to perform the desired integration with high accuracy. For some values of τ the integration may also be faster than standard integration techniques.

We begin by considering what values of θ give $|n\theta - k| \leqslant \tau$. We have $|\theta - k/n| \leqslant \tau/n$; that is, θ is in the interval $[k/n - \tau/n, k/n + \tau/n] = I(k/n)$ about k/n. Hence if $\theta \in I(k/n)$ and $\theta \notin I(i/j)$ for any $j \leqslant n$ with $i/j \neq k/n$, then $|n\theta - k| \leqslant \tau$ represents the primary overlap in the sense of Section 2. Clearly, $(n, k) = 1$. We also note that it is easily seen that n must be less that $1/\tau$. Let N be the least integer $\geqslant 1/\tau - 1$; we are thus led to consider the set of fractions k/n where $(n, k) = 1$ and

$n \leqslant N$. It will be convenient for the moment to let k assume positive or negative values.

If these fractions are arranged in ascending order, the resulting finite sequence is called a Farey series of order N, often written F_N. The properties of Farey series are well known; for instance, see [2, pp. 141–144]. We will use a number of properties of F_N proved in [2]. Of course, negative terms do not have practical significance here. If k/n and k'/n' are two consecutive terms of F_x, we call the fraction $(k + k')/(n + n')$ the mediant. We now state

Property I. Let F_x be given. Between every pair of consecutive terms k/n and k'/n' for which $n + n' = x + 1$, insert their mediant. Then the resulting sequence is F_{x+1}.

The other properties can be easily derived from the first.

Property II. If k/n and k'/n' are two consecutive terms of F_x, then $nk' - n'k = 1$.

Property III. If k/n and k'/n' are two consecutive terms of F_x, then $n + n' \geqslant x + 1$.

Property IV. If $x \geqslant 2$, then no two consecutive terms of F_x have the same denominator.

We will determine the manner in which the intervals $I(k/n)$ overlap. Let k/n and k'/n' be two consecutive terms of F_N. The right-hand endpoint of $I(k/n)$ is $(k + \tau)/n$, and the left-hand endpoint of $I(k'/n')$ is $(k' - \tau)/n'$. A simple calculation using Properties II and III shows that $I(k/n)$ and $I(k'/n')$ overlap.

On the other hand, using Property II and the fact that $N < 1/\tau$, it can be seen that neither $I(k/n)$ nor $I(k'/n')$ overlaps the center of the other. Hence, about each k/n in F_N there is a nontrivial interval $J(k/n) \subset I(k/n)$ for which $|n\theta - k| \leqslant \tau$ represents the primary overlap. We will call these $J(k/n)$ the primary intervals. If k'/n', k/n, and k''/n'' are three consecutive terms of F_N, the left-hand endpoint of $J(k/n)$ is $(k - \tau)/n$ if $n < n'$ and is $(k' + \tau)/n'$ if $n' < n$, since in the region common to $I(k'/n')$ and $I(k/n)$ the primary overlap occurs at $\min(n, n')$. We cannot have $n = n'$ by Property IV and the fact that $\tau < ½$. Likewise, the right-hand endpoint of $J(k/n)$ is $(k + \tau)/n$ if $n < n''$ and is $(k'' - \tau)/n''$ if $n > n''$.

We now must determine within each primary interval $J(k/n)$ subintervals within which the secondary overlap occurs at m. We first divide $J(k/n)$ into two parts: $J_1(k/n)$, for which $\theta \geqslant k/n$, and $J_2(k/n)$, for which $\theta \leqslant k/n$. (As usual, we are not really concerned with $k/n = \theta$, since this case will normally have probability zero.) We consider $J_1(k/n)$ first. Since $0 \leqslant n\theta - k \leqslant \tau$, we seek the smallest m for which, for some l, $-\tau \leqslant m\theta - l < 0$. For the moment let $n > 1$ and let i'/j' and i/j be respectively the left-hand and right-hand neighbors of k/n in F_n, so that k/n is the mediant of i'/j' and i/j. (Note that they are neighbors in F_n, not necessarily in F_N.) Certainly $k/n < \theta < i/j$. We assume the contrary, that for some x and y with

$x < n$, $|x\theta - y|$ is minimized, and that $x \neq j$. By a result of Section 2, $\theta - y/x < 0$. Also, since $x < n$, y/x is a term of F_n; thus y/x is to the right of i/j. Hence $y/x - \theta > y/x - i/j = (yj - ix)/jx \geqslant 1/jx$, so that $y\theta - x < -1/j$. On the other hand,

$$i/j - \theta \leqslant i/j - k/n = (in - kj)/jn = 1/jn,$$

so $j\theta - i \geqslant -1/n$. But $j < n$, so $y\theta - x < j\theta - i < 0$, which is a contradiction. Returning to the case $n = 1$, we see that we may use the convention $i/j = 1/0$.

Since $|j\theta - i|$ is minimal, we have from Section 2 that the m and l determining the secondary overlap have the form $Kn + j$ and $Kk + i$ respectively, where $K \geqslant 1$. The condition that m and l determine the secondary overlap is easily seen to be

$$-\tau \leqslant m\theta - l < -\tau + \tau_1.$$

We therefore have $-\tau \leqslant m\theta - l < -\tau + n\theta - k$. Thus

$$\frac{l - \tau}{m} \leqslant \theta < \frac{l - k - \tau}{m - n}.$$

Expressing this in terms of K, we have

$$\frac{Kk + i - \tau}{Kn + j} \leqslant \theta < \frac{(K - 1)k + i - \tau}{(k - 1) + j}.$$

We will designate such an interval by $I_1(k/n, K)$. Of course, it will often happen that some of these intervals are not contained in, or are only partially contained in, $J_1(k/n)$. We wish to study the manner in which this happens.

We see that each interval is adjacent to the previous one and that they decrease as K increases, having k/n as a limit. Thus, if K_0 is the smallest integer for which the left-hand endpoint of $I_1(k/n, K_0)$ is less than the right-hand endpoint of $J_1(k/n)$, then $I_1(k/n, K) \subset J_1(k/n)$ if $K > K_0$, $I_1(k/n, K) \cap J_1(k/n) = \varnothing$ if $K < K_0$, and $I_1(k/n, K_0)$ may in general be only partially contained in $J_1(k/n)$. Hence, we wish to find K_0. We distinguish two cases. First, let $I(i/j)$ overlap $I(k/n)$, so that the right-hand endpoint of $J_1(k/n)$ is $(i - \tau)/j$. The condition for this to occur is easily seen to be $n + j > N$. But this is also the right-hand endpoint of $I_1(k/n, 1)$, so that $K_0 = 1$ and the entire interval $I_1(k/n, 1)$ is in $J_1(k/n)$. Second, let $I(i/j)$ and $I(k/n)$ not overlap, so the right-hand endpoint of $J_1(k/n)$ is $(k + \tau)/n$. Then, we must have

$$\frac{K_0 k + i - \tau}{K_0 n + j} \leqslant \frac{k + \tau}{n};$$

$$K_0 kn + ni - n\tau \leqslant K_0 nk + K_0 n\tau + jk + j\tau;$$

$$K_0 n\tau \geqslant ni - jk - (n + j)\tau = 1 - (n + j)\tau;$$

$$K_0 \geqslant (1 - (n + j)\tau)/n\tau.$$

Thus K_0 is the least integer greater than or equal to the right-hand expression. We will designate by $I_1'(k/n, K_0)$ the interval defined by

$$\frac{K_0 k + i - \tau}{K_0 n + j} \leqslant \theta \leqslant \frac{k+\tau}{n}.$$

The analysis for the second case, namely $\theta \leqslant k/n$, is essentially the same. The subintervals of interest in this case, designated by $I_2(k/n, K)$, are defined by

$$\frac{(K-1)k + i' + \tau}{(K-1)n + j'} < \theta \leqslant \frac{Kk + i' + \tau}{Kn + j'}.$$

If $n = 1$, we use the convention $i'/j' = -1/0$. As before, if $I(i'/j')$ overlaps $I(k/n)$, $K_0 = 1$ and $I_2(k/n, 1)$ is contained in $J_2(k/n)$. If not, the expression for K_0 remains the same as before, and $I_2'(k/n, K_0)$ becomes the interval defined by

$$\frac{k-\tau}{n} \leqslant \theta \leqslant \frac{K_0 k + i' + \tau}{K_0 n + j'}.$$

We are now in a position to integrate $P_\theta(s)$ in each interval $J_1(k/n)$ or $J_2(k/n)$ and thus to calculate $P(s)$. As usual, we consider $J_1(k/n)$ first. We know, from Section 2, that if $\theta \in J_1(k/n) \cap I_1(k/n, K)$, then $P_\theta(s)$ is given by Theorem 1. But $m = Kn + j$, $\tau_1 = n\theta - k$, $1 + jk = in$, and some calculation leads to

$$\tau_2 = ((K-1)k + i) - ((K-1)n + j)\theta - \tau.$$

Thus, we may calculate the integral of $P_\theta(s)$ over the interval $I_1(k/n, K) \subset J_1(k/n)$:

$$\int_{I_1(k/n,K)} P_\theta(s) \cdot R'(\theta)d\theta = \begin{cases} \tau \int_{I_1(k/n,K)} R'(\theta)\,d\theta & \text{if } 0 \leqslant s \leqslant n-1, \\ \int_{I_1(k/n,K)} (n\theta - k)R'(\theta)d\theta & \text{if } n \leqslant s \leqslant Kn + j - 1, \\ \int_{I_1(k/n,K)} (((K-1)k + i) - ((K-1)n + j)\theta - \tau)R'(\theta)\,d\theta & \\ \qquad\qquad \text{if } Kn + j \leqslant s \leqslant (K+1)n + j - 1, & \\ 0 \text{ if } s \geqslant (K+1)n + j. & \end{cases}$$

The above expressions also hold for the interval $I_1'(k/n, K_0)$, if applicable.

Set $S(\theta) = \int \theta R'(\theta)\,d\theta$, the indefinite integral of $\theta R'(\theta)$, and let the endpoints of $I_1(k/n, K)$ or $I_1'(k/n, K)$ be α and β, $\alpha < \beta$. Then we may write the above as

$$\int_\alpha^\beta P_\theta(s)\cdot R'(\theta)d\theta = \begin{cases} \tau(R(\beta)-R(\alpha)) \text{ if } 0\leqslant s\leqslant n-1, \\ n(S(\beta)-S(\alpha))-k(R(\beta)-R(\alpha)) \text{ if } n\leqslant s\leqslant Kn+j-1, \\ ((K-1)k+i-\tau)(R(\beta)-R(\alpha))-((K-1)n+j)(S(\beta)-S(\alpha)) \\ \qquad\qquad \text{if } Kn+j\leqslant s\leqslant (K+1)n+j-1, \\ 0 \text{ if } s\geqslant (K+1)n+j. \end{cases}$$

We could use these formulas to calculate the values for each subinterval and sum them all to get the desired values of $P(s)$; however, we can simplify the calculations by combining the integrals over sets of subintervals. We will let α_K and α_{K+1}, with $\alpha_{K+1} < \alpha_K$, be the endpoints of the interval $I_1(k/n, K)$. Where we have replaced $I_1(k/n, K_0)$ by $I_1'(k/n, K_0)$, we will let α_{K_0} and α_{K_0+1} designate the endpoints of $I_1'(k/n, K_0)$. We will calculate, for each t,

$$\int_{J_1(k/n)} P_\theta(s)R'(\theta)d\theta = \int_{k/n}^{\alpha_{K_0}} P_\theta(s)R'(\theta)d\theta.$$

We first observe that if $0 \leqslant t < n$, this integral has the value

$$\tag{1} \begin{aligned} &\tau(R(\alpha_{K_0})-R(\alpha_{K_0+1}))+\tau(R(\alpha_{K_0+1})-R(\alpha_{K_0+2}))+\cdots \\ &\qquad = \tau(R(\alpha_{K_0})-R(k/n)) = \tau R(k/n, \alpha_{K_0}), \end{aligned}$$

where $R(\theta_1, \theta_2)$ and $S(\theta_1, \theta_2)$ denote $R(\theta_2)-R(\theta_1)$ and $S(\theta_2)-S(\theta_1)$, respectively. Next, we see that if $n \leqslant s < K_0 n + j$, the expression for the integral is the second one appearing above for all subintervals. Hence, in this case, making use of similar cancellations,

$$\tag{2} \int_{J_1(k/n)} P_\theta(s)R'(\theta)d\theta = nS(k/n, \alpha_{K_0}) - kR(k/n, \alpha_{K_0}).$$

Finally, for each $K \geqslant K_0$ consider the range of values $Kn+j \leqslant s < (K+1)n+j$. The subintervals (if any) with index less than K give an integral of zero; the expression for the integral over subinterval K is the third appearing above; and the expression for the integral over all subintervals with index greater than K is the second appearing above. Hence, in this case we are led to

$$\tag{3} \begin{aligned} \int_{J_1(k/n)} P_\theta(s)R'(\theta)\,d\theta &= ((K-1)k+i-\tau)R(k/n, \alpha_K)-(Kk+i-\tau)R(k/n, \alpha_{K+1}) \\ &\quad -((K-1)n+j)S(k/n, \alpha_K)+(Kn+j)S(k/n, \alpha_{K+1}). \end{aligned}$$

We now consider the integral over $J_2(k/n)$. The formulas for the integral over $I_2(k/n, K)$ become

$$\int_{\alpha}^{\beta} P_\theta(s) R'(\theta) d\theta = \begin{cases} \tau(R(\beta) - R(\alpha)) & \text{if } 0 \leqslant s \leqslant n-1, \\ k(R(\beta) - R(\alpha)) - n(S(\beta) - S(\alpha)) & \text{if } n \leqslant s \leqslant Kn + j' - 1, \\ ((K-1)n + j')(S(\beta) - S(\alpha)) - ((K-1)k + i' + \tau)(R(\beta) - R(\alpha)) & \\ \qquad \text{if } Kn + j' \leqslant s \leqslant (K+1)n + j' - 1, & \\ 0 & \text{if } s \geqslant (K+1)n + j'. \end{cases}$$

We let β_K and β_{K+1} be the endpoints of $I_2(k/n, K)$ or $I_2'(k/n, K)$, with $\beta_K < \beta_{K+1}$ and with K_0' in place of K_0. The integrals over $J_2(k/n)$ become

$$\int_{J_2(k/n)} P_\theta(s) R'(\theta) d\theta = \begin{cases} \tau R(\beta_{K_0}, k/n) \quad \text{if } 0 \leqslant s < n, & (4) \\ kR(\beta_{K_0}, k/n) - nS(\beta_{K_0}, k/n) \quad \text{if } n \leqslant s < K_0' n + j', & (5) \\ ((K-1)n + j')S(\beta_K, k/n) - (Kn + j')S(\beta_{K+1}, k/n) & \\ - ((K-1)k + i' + \tau)R(\beta_K, k/n) + (Kk + i' + \tau)R(\beta_{K+1}, k/n) & (6) \\ \qquad \text{if } Kn + j' \leqslant s < (K+1)n + j', K \geqslant K_0'. & \end{cases}$$

Formulas (1)–(6) provide the basis for calculating $P(s)$, and at this point the number-theoretic work necessary is complete. In principle, all that is now needed is to sum the calculated values over all $k/n \in \mathfrak{F}_N$. However, to use these formulas effectively, it is necessary to study the computational aspects of the problem. This will be the task of [1].

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