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A Combinatorial Theorem in Group Theory*

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To D. H. Lehmer on his 70th birthday

Abstract. There is an anti-Ramsey theorem for inhomogeneous linear equations over a field, which is essentially due to R. Rado [2]. This theorem is generalized to groups to get sharper quantitative and qualitative results. For example, it is shown that for any Abelian group A (written additively) and any mappings $f_1, \cdots, f_n$ of A into itself there exists a k-coloring χ of A so that the inhomogeneous equation

$$\sum_{i=1}^{n} (f_i(x_i) - f_i(y_i)) = b, \quad b \neq 0$$

has no solutions x_i, y_i with $\chi(x_i) = \chi(y_i)$ for all $i = 1, \cdots, n$. Here the number of colors k can be chosen bounded by $(3n)^{n-1}$ which depends on n alone and not on the f_i or b. For non-Abelian groups an analogous qualitative result is proven when b is "residually compact". Applications to anti-Ramsey results in Euclidean geometry are given.

1. Introduction. Richard Rado [2] has shown that for certain fields F it is possible to color the elements of F in a finite number of colors so that an inhomogeneous equation

$$\sum_{i=1}^{n} a_i x_i = b; \quad a_i, b \in F, b \neq 0, \sum a_i = 0 \tag{1.1}$$

has no solution $(x_1, \cdots, x_n)$ where all x_i have the same color. This result was sharpened and extended to all fields F in [1] where it was used to prove that for any set S in a Euclidean space E^n, with the property that S cannot be isometrically embedded on a sphere, there is a finite coloring of Hilbert space H so that no set isometric to S has all its points of the same color. Sharper results involving the minimal number of different colored points in sets isometric to S were also obtained in [1].

In this note we recognize Rado's coloring problem to be essentially group theoretic in nature. This enables us to get a simpler and more general theorem than [1, Theorem 16] and to get a much better and more uniform estimate on the number of colors needed to prevent the existence of a monochromatic copy of a nonspherical set S of E^n in H.

2. Formulation of Problems and Definitions. Let G be a group and $f_1, f_2, \cdots, f_n$ arbitrary mappings of G into G. Let $b \in G$, $b \neq 1$.

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Problem 2.1. Under what conditions is there a finite coloring χ of G, using k colors, so that the equation

$$f_1(x_1)f_1(y_1)^{-1}f_2(x_2)f_2(y_2)^{-1}\cdots f_n(x_n)f_n(y_n)^{-1} = b \tag{2.2}$$

has no solution $x_1,\cdots,x_n,y_1,\cdots,y_n \in G$ with $\chi(x_i) = \chi(y_i)$ for all $i = 1, 2, \cdots, n$?

Note that the inhomogeneity condition, $b \neq 1$, is essential, since otherwise we would always have a system of solutions $x_i = y_i$. Also we are not asking merely to prevent solutions where all the variables on the left of (2.2) have the same color, but all solutions where only the coupled pairs of variables x_i, y_i have the same color.

Problem 2.3. If Problem 2.1 can be answered in the affirmative what can be said about the minimal number, k, of colors needed? In particular, to what extent does k depend on the data $n, b, f_1, \cdots, f_n$?

Let P be any property of groups which is invariant under isomorphisms (that is P defines a class of groups). We shall use this concept somewhat informally, such as the property "G is compact" to mean that G can be given a group topology under which it is compact.

Definition 2.4. An element $b \in G$ is ***residually*** P if there is a homomorphism, φ, of G into a group $\bar{G}$ with property P so that $\varphi(b) \neq 1$.

The group G is ***residually*** P if all its nonidentity elements are residually P.

The term residually finite is a familiar one, and, for example, all free groups are residually finite. An element of G is residually Abelian if and only if it is not an element of the commutator subgroup G'.

3. Qualitative Results.

THEOREM 3.1. *Let b be a residually compact element of G; then Problem 2.1 has an affirmative solution and the number of colors, k, has a bound which is independent of the functions $f_1, \cdots, f_n$.*

Proof. Let φ be a homomorphism of G into the compact group $\bar{G}$ so that $\bar{b} = \varphi(b) \neq 1$.

We now color $\bar{G}$ by a coloring $\bar{\chi}$ as follows: Pick a neighborhood U of the identity in $\bar{G}$ so that $U = U^{-1}$ and $b \notin U^n$. Let $U_i = U\bar{g}_i$; $i = 1, \cdots, N$, be a finite covering of $\bar{G}$ by translates of U. Now define $\bar{\chi}(\bar{x})$ to be the least index i for which $\bar{x} \in U_i$. Thus the equation

$$\prod_{i=1}^{n} \bar{x}_i \bar{y}_i^{-1} = \bar{b} \tag{3.2}$$

has no solution with $\bar{\chi}(\bar{x}_i) = \bar{\chi}(\bar{y}_i)$, $i = 1, \cdots, n$, since that would imply $\bar{x}_i\bar{y}_i^{-1} \in U$ and $\bar{b} \in U^n$ contrary to hypothesis. It is clear that the number $\bar{k}$ of colors in $\bar{\chi}$ depends only on n and $\bar{b}$.

Now define the coloring χ on G by

$$\chi(x) = \chi(y) \Leftrightarrow \overline{\chi}(\overline{f_i(x)}) = \overline{\chi}(\overline{f_i(y)}) \qquad (i = 1, \cdots, n). \tag{3.3}$$

Then applying the homomorphism φ to a solution of (2.2) we get

$$\prod_{i=1}^{n} \overline{f_i(x_i)}\,\overline{f_i(y_i)}^{-1} = \overline{b},$$

which has no solution with $\overline{\chi}(\overline{f_i(x_i)}) = \overline{\chi}(\overline{f_i(y_i)})$ $(i = 1, \cdots, n)$ and hence (2.2) has no solution with $\chi(x_i) = \chi(y_i)$ $(i = 1, \cdots, n)$.

The number, k, of colors in χ is no greater than $\overline{k}^m$, where m is the number of distinct functions among $f_1, \cdots, f_n$. In any case we have $k \leqslant \overline{k}^n$ where the number on the right is independent of $f_1, \cdots, f_n$.

We have already mentioned that Theorem 3.1 applies to all $b \neq 1$ in a free group, since such groups are residually finite and hence a fortiori residually compact. Abelian groups, A, are in general not residually finite, but they are residually compact. In fact for every $b \in A$, $b \neq 0$, there is a mapping φ of A into the circle group with $\varphi(b) \neq 0$.

THEOREM 3.4. *Let A be an Abelian group $b \in A$, $b \neq 0$. Let B be a maximal subgroup of $A \setminus \{b\}$. Then*

$$A/B \cong Z_{p^n}, \qquad n = 1, 2, \cdots, \text{or } \infty, \tag{3.5}$$

where Z_{p^n} denotes the multiplicative group of all p^nth roots of unity and $Z_{p^\infty} = \bigcup_n Z_{p^n}$.

The prime p in (3.5) can be chosen to be any divisor of ord b, *by a suitable choice of B. In particular, if* ord $b = \infty$, *then p can be chosen arbitrarily.*

Proof. In the group $\overline{A} = A/B$ every nontrivial subgroup must contain the element $\overline{b} = b + B$. Thus $\overline{b}$ must be of prime order p and $\overline{A}$ is a p-group in which every finitely generated subgroup is cyclic. Thus, if $\overline{A}$ is finite, it is cyclic of order p^n; and if $\overline{A}$ is infinite it is isomorphic to Z_{p^∞}.

It is clear that for any p which divides ord b we can choose B to contain all elements whose order is prime to p as well as the element pb, so that $\overline{A}$ is a p-group.

In [1] we constructed an example to show that we cannot extend theorems of Rado type to cases where Eq. (1.1) is replaced on the left by a homogeneous form of higher degree. The reason for this situation is now clearer, since the nonzero elements in the additive group of rationals are residually compact while the analogous result does not hold for the ring of rational numbers. We can state an example of the general algebraic situation as follows.

THEOREM 3.6. *Let A be an algebraic system with a number of operations. Let $F(x_1, \cdots, x_n)$ be an expression in A with variables $x_1, \cdots, x_n$ so that $F(x, \cdots, x) = a$ for all $x \in \mathsf{A}$. Assume there is a $b \in \mathsf{A}$ and a homomorphism φ of A into an algebra A with a compact topology in which all operations are continuous and $\overline{b} = \varphi(b) \neq \overline{a} = \varphi(a)$.*

Then for any set of mappings $f_1, \cdots, f_n$ *of* A *into* A *there exists a finite coloring* χ *of* A *so that the equation*

$$F(f_1(x_1), \cdots, f_n(x_n)) = b \tag{3.7}$$

has no solution with $\chi(x_1) = \cdots = \chi(x_n)$.

Proof. It is clear that under the homomorphism φ the expression F becomes $\bar{F}(\bar{x}_1, \cdots, \bar{x}_n)$ with $\bar{F}(\bar{x}, \cdots, \bar{x}) = \bar{a}$ for all $\bar{x} \in \bar{A}$. By continuity there is a neighborhood $U(\bar{x})$ for each $\bar{x} \in \bar{A}$ so that $F(\bar{x}_1, \cdots, \bar{x}_n) \neq \bar{b}$ for all $\bar{x}_1, \cdots, \bar{x}_n \in U(\bar{x})$. By the compactness of $\bar{A}$ there is a finite set $\{U_1, \cdots, U_N\}$ of these neighborhoods which cover $\bar{A}$ and we define the coloring $\bar{\chi}$ of $\bar{A}$ by $\bar{\chi}(\bar{x}) = \bar{\chi}(\bar{y})$ if and only if $\bar{x}$ and $\bar{y}$ belong to the same elements of $\{U_1, \cdots, U_N\}$. Finally we again define the coloring χ of A by $\chi(x) = \chi(y)$ if and only if $\bar{\chi}(\overline{f_i(x)}) = \bar{\chi}(\overline{f_i(y)})$ for $i = 1, \cdots, n$.

COROLLARY 3.8. *Let* R *be a ring and let* M *be a maximal ideal so that* R/M *is finite. If* $b \notin M$, *then for any homogeneous form*

$$F(x_1, \cdots, x_n) = \sum_{i_1=1}^{n} \sum_{i_k=1}^{n} a_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k}$$

there is a finite coloring χ *of* R *so that the equation* $F(x_1 - y_1, \cdots, x_n - y_n) = b$ *has no solution with* $\chi(x_i) = \chi(y_i)$, $i = 1, \cdots, n$.

I have not been able to prove that the converse of Theorem 3.1 holds, however we can get a result which comes close to such a converse.

THEOREM 3.9. *Let* G *be a group and* $b \in G$ *such that for every positive integer* n *there exists a finite coloring* χ *of* G *so that the equation*

$$\prod_{i=1}^{n} x_i y_i^{-1} = b$$

has no solution with $\chi(x_i) = \chi(y_i)$, $i = 1, \cdots, n$.

Then there exists a homomorphism φ *of* G *onto a group* $\bar{G}$ *where* $\bar{G}$ *contains a family* $\mathcal{S}$ *of subsets with the following properties*:

(i) *Each* $S \in \mathcal{S}$ *has a finite number of translates* $S\bar{g}_1, \cdots, S\bar{g}_N$ *(*N *depending on* S*) so that* $S\bar{g}_1 \cup \cdots \cup S\bar{g}_N = \bar{G}$.

(ii) *For each positive integer* n *there is an* $S \in \mathcal{S}$ *so that* $\bar{b} \notin S^n$.

(iii) $S = S^{-1}$ *for all* $S \in \mathcal{S}$.

(iv) $\bigcap_{\mathcal{S}} S = \{1\}$.

Proof. We first note that, given any automorphism α of G and any positive integer n there exists a finite coloring χ_α of G so that the equation

$$\prod_{i=1}^{n} x_i y_i^{-1} = \alpha(b)$$

has no solution with $\chi_\alpha(x_i) = \chi_\alpha(y_i)$, $i = 1, \cdots, n$. This coloring is obtained by $\chi_\alpha(x) = \chi(\alpha^{-1}(x))$. By superposition of a finite number of such colorings we get the result that for any finite set A of automorphisms of G and any positive integer n there exists a finite coloring χ_A of G so that

$$\prod_{i=1}^{n} x_i y_i^{-1} \neq \alpha(b) \quad \text{for any } \alpha \in A \tag{3.10}$$

whenever $\chi_A(x_i) = \chi_A(y_i)$, $i = 1, \cdots, n$.

We now restrict attention to finite sets A of inner automorphisms and construct finite colorings $\chi_{n,A}$ where χ_{n_2,A_2} is a refinement of χ_{n_1,A_1} whenever $n_2 \geqslant n_1$, $A_2 \supseteq A_1$; and each $\chi_{n,A}$ prevents a solution of (3.10) with $\chi_{n,A}(x_i) = \chi_{n,A}(y_i)$. To each χ we associate the set

$$S(n, A) = \{xy^{-1} \mid \chi_{n,A}(x) = \chi_{n,A}(y)\},$$

which we could regard as the set of elements whose color is that of the identity. These sets are ordered by inclusion $S(n_2, A_2) \subseteq S(n_1, A_1)$ if $n_2 \geqslant n_1$, $A_2 \supseteq A_1$. They all satisfy $S = S^{-1}$ and each set has a finite number of translates which cover G. To see this let $g_1, \cdots, g_N$ represent the different colors of $\chi_{n,A}$. Then $S = S(n, A)$ satisfies $Sg_1 \cup \cdots \cup Sg_N = G$. By hypothesis $b \notin (S(n, A))^n$.

Finally, if $\alpha \in A$ then our construction yields

$$\alpha^{-1}(S(n, A)) \subset S(n, A \setminus \{\alpha\}).$$

Let $S_0 = \bigcap S(n, A)$; then S_0 is closed under inner automorphisms of G and the normal subgroup G_0 generated by S_0 does not contain b, since no finite product of elements of S_0 is equal to b.

The natural homomorphism $G \longrightarrow G/G_0 = \overline{G}$ maps the sets $S(n, A)$ onto a family of sets S with the desired properties.

4. Quantitative Results. For Abelian groups, and hence for residually Abelian elements in arbitrary groups, Theorem 3.4 gives us quantitative results which we can apply to Euclidean anti-Ramsey results.

THEOREM 4.1. *Let A be an Abelian group and let $f_1, \cdots, f_n$ be mappings of A into A with m ($\leqslant n$) the number of distinct mappings. Let $b \in A^*$; then there exists a k-coloring χ of A so that the equation*

$$\sum_{i=1}^{n} (f_i(x_i) - f_i(y_i)) = b$$

has no solutions with $\chi(x_i) = \chi(y_i)$, where

$$k = \begin{cases} (2n)^m & \textit{if } 2 \mid \text{ord } b \textit{ or } \text{ord } b = \infty, \\ \left\lceil \dfrac{2np}{p-1} \right\rceil^m & \textit{where } p \textit{ is the largest prime divisor of } \text{ord } b, \\ & \textit{if } \text{ord } b \textit{ is odd.} \end{cases}$$

Here $\lceil x \rceil$ *denotes the smallest integer* $\geqslant x$.

Proof. We perform the homomorphism $A \longrightarrow A/B$ used in the proof of Theorem 3.4 and choose the prime $p = 2$ if $\operatorname{ord} b$ is even or infinite and the largest prime divisor p of $\operatorname{ord} b$ if $\operatorname{ord} b$ is odd. Theorem 3.4 shows that $\bar{A} = A/B$ is isomorphic to a subgroup of the circle group, or, equivalent of the additive group of rationals (mod 1). By a suitable automorphism we may choose $\bar{b}$ to be any element of order p in $\bar{A}$ and we therefore choose $\bar{b} = \frac{1}{2}$ if $p = 2$ and $\bar{b} = (p-1)/(2p)$ if $p > 2$.

If we $\bar{k}$-color the interval $[0, 1)$ by $\bar{\chi}$ so that each interval $[(i-1)/(2n), i/(2n))$, $i = 1, 2, \cdots, 2n$, represents one color when $p = 2$, and so that $[(i-1)(p-1)/(2pn)$, $\min\{i(p-1)/(2pn), 1\})$, $i = 1, 2, \cdots, \lceil 2np(p-1) \rceil$, represents one color when $p > 2$ then $\bar{\chi}(\bar{x}) = \bar{\chi}(\bar{y})$ implies $|\bar{x} - \bar{y}| < 1/(2n)$ and $|\bar{x} - \bar{y}| < (p-1)/(2pn)$, respectively, so that the equation

$$\sum_{i=1}^{n} (\bar{x}_i - \bar{y}_i) = \bar{b}$$

has no solutions with $\bar{\chi}(\bar{x}_i) = \bar{\chi}(\bar{y}_i)$, $i = 1, \cdots, n$, and $\bar{k} = 2n$ or $\lceil 2np/(p-1) \rceil$.

Finally we define the coloring χ of A by $\chi(x) = \chi(y)$ if and only if $\bar{\chi}(\overline{f_i(x)}) = \bar{\chi}(\overline{f_i(y)})$, $i = 1, \cdots, n$, so that χ is the desired k-coloring with $k = \bar{k}^m$.

In view of Theorems 13 and 25 of [1] we can now state the following results.

COROLLARY 4.2. *Given a set* S *of* n *points which cannot be embedded in a sphere, there exists a* $(2(n-1))^{n-1}$*-coloring* χ *of Hilbert space* H *by concentric spheres* ($\chi(x)$ *depends only on the norm* $\|x\|$) *so that no set congruent to* S *in* H *is monochromatic.*

COROLLARY 4.3. *Given a set* S *of* n *points which cannot be embedded in fewer than* l *concentric spheres, there exists a sphere-coloring* χ *of Hilbert space using no more than*

$$k = (2(n - l + 1))^{(n-l+1)S(n,l-1)}$$

colors, where $S(n, l-1)$ *is the Stirling number, so that every set congruent to* S *in* H *has at least* l *distinct colors.*

Proof. There are $S(n, l-1)$ partitions of S into $l-1$ nonempty sets S_1, $\cdots, S_{l-1}$. Since these sets cannot lie on spheres with a common center, it follows from [1, Lemma 26] that there is an equation

$$\sum_{j=1}^{l-1} \sum_{i=1}^{|S_j|-1} c_{ij} (\|x_{ij}\|^2 - \|x_{i0}\|^2) = b \neq 0 \tag{4.4}$$

which would have to have solutions $\chi(\|x_{ij}\|^2) = \chi(\|x_{i0}\|^2)$ for a sphere coloring in which each S_j is monochromatic. The number of terms on the left of (4.4) is $\Sigma(|S_j| - 1) = n - l + 1$ so that by Theorem 4.1 there is a $(2(n-l+1))^{n-l+1}$-coloring which prevents such a solution of (4.4). We get our result by superposition of the $S(n, l-1)$ different colorings.

For example if we have 4 collinear points without a center of symmetry, then there exists a 16^7-coloring of H in which every congruent set has at least 3 colors.

Theorem 4.1 shows that we can get bounds on the number of colors needed to prevent monochromatic solutions which depend only on the number n of summands and not on the mappings f_i or the element b. We have already shown by the proof of [1, Theorem 17] that this dependence on n cannot be removed. It might be interesting to get precise quantitative results. As an example we give the following

THEOREM 4.5. *Let χ be a coloring of the additive group of integers and let $N \leqslant n$ be the least common multiple of the first k integers. If*

$$\sum_{i=1}^{n} (x_i - y_i) = N$$

has no solution with $\chi(x_i) = \chi(y_i)$, $i = 1, \cdots, n$, then χ has at least $k+1$ distinct colors.

Proof. If any two of the numbers in $\{0, 1, \cdots, k\}$ have the same color, then we can pick x, y so that $\chi(x) = \chi(y)$ and $0 < x - y \leqslant k$. Thus $N = m(x-y)$ with $m \leqslant n$. This gives the rather weak result that the number of colors must go to infinity at least as rapidly as $\log n$.

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1. P. ERDÖS, R. L. GRAHAM, P. MONTGOMERY, B. L. ROTHSCHILD, J. SPENCER & E. G. STRAUS, "Euclidean Ramsey theorems. I," *J. Combinatorial Theory Ser. A*, v. 14, 1973, pp. 341–363. MR 47 #4825.

2. R. RADO, "Note on combinatorial analysis," *Proc. London Math. Soc.* (2), v. 48, 1943, pp. 122–160. MR 5, 87.