MATHEMATICS OF COMPUTATION, VOLUME 29, NUMBER 130
APRIL 1975, PAGES 343–349

Error Estimates for a Finite Element Approximation of a Minimal Surface

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Abstract. A finite element approximation of the minimal surface problem for a strictly convex bounded plane domain Ω is considered. The approximating functions are continuous and piecewise linear on a triangulation of Ω. Error estimates of the form $O(h)$ in the H^1 norm and $O(h^2)$ in the L_p-norm ($p < 2$) are proved, where h denotes the maximal side in the triangulation.

1. Introduction. Let Ω be a strictly convex bounded domain in the plane R^2 with smooth (two times continuously differentiable, say) boundary Γ, and let φ be a given function defined on Γ. Consider the following minimal surface problem: Find a function u which minimizes the integral

$$\int_\Omega \sqrt{1 + |\nabla v|^2}\, dx, \qquad \nabla v = \text{grad } v,$$

over all Lipschitz functions v in Ω such that $v = \varphi$ on Γ. It is known (see, e.g., [2, Theorem 4.2.1]) that if φ is the restriction to Γ of a function in the Sobolev space $W_q^2(\Omega)$ for some $q > 2$, and if φ satisfies the bounded slope condition (see [2]), then there is a unique minimizing function $u \in W_q^2(\Omega)$.

For the purpose of the approximate solution of this problem, for each h with $0 < h < 1$, let $T_h = \{T_j\}$ be a finite collection of closed triangles T_j such that $\Omega \subset \bigcup_j T_j$, and such that any T_j with $T_j \cap \Omega \neq \emptyset$ is either contained in $\overline{\Omega}$ or has two vertices on Γ. It is also assumed that the triangles have disjoint interiors, that no vertex of any triangle is on the interior of an edge of another triangle, and that there is a constant c, with $0 < c < 1$ independent of h, such that the edges of the triangles have length between ch and h, and all angles of the triangles are bounded below by c. Denoting the union of the triangles contained in $\overline{\Omega}$ by Ω_h, we let S_h be the set of continuous functions defined on Ω_h which are linear on each T_j and assume the same values as φ on the vertices of the triangulation on Γ. Consider now the following finite element method for the approximate solution of the given problem: Find a function u_h which minimizes the integral $\int_{\Omega_h} \sqrt{1 + |\nabla v_h|^2}\, dx$ over all functions $v_h \in S_h$. To see that there exists a unique minimizing function u_h, we notice that the function

$$f(y) = \sqrt{1 + |y|^2}, \qquad y = (y_1, y_2) \in R^2, \qquad |y|^2 = y_1^2 + y_2^2,$$

Received March 22, 1974.
AMS (MOS) subject classifications (1970). Primary 65N10, 65N30.

is strictly convex since, with $f_{,ij} = \partial^2 f/\partial y_i \partial y_j$,

$$
\begin{aligned}
f_{,ij}(y)\xi_i\xi_j &= (1 + |y|^2)^{-3/2}[(1 + y_2^2)\xi_1^2 - 2y_1y_2\xi_1\xi_2 + (1 + y_1^2)\xi_2^2] \\
&\geqslant (1 + |y|^2)^{-3/2}|\xi|^2 \quad \text{for } \xi \in R^2.
\end{aligned}
\tag{1.1}
$$

Here and below, we use the summation convention; repetition of an index i indicates summation over $i = 1, 2$. Since f is strictly convex, the mapping F: $v_h \longrightarrow \int_{\Omega_h} f(\nabla v_h)dx$, $v_h \in S_h$, is also strictly convex. Furthermore, it is clear that $F(v_h)$ tends to infinity with $\max_{\Omega_h}|v_h|$. Since F is continuous and S_h is finite dimensional, it then follows easily that there exists a unique minimizing function u_h.

In this note, we shall prove some convergence estimates for the finite element method described above. In order to express our results, we introduce for k an integer, $1 \leqslant p \leqslant \infty$, the following (semi) norms:

$$
|v|_{k,p} = \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha v|^p \, dx\right)^{1/p}, \qquad \|v\|_{k,p} = \left(\sum_{j \leqslant k} |v|_{k,p}^p\right)^{1/p},
$$

with the usual modification if $p = \infty$. We shall also need corresponding norms with Ω replaced by Ω_h, and we shall then use the notation $|\cdot|_{k,p,h}$ and $\|\cdot\|_{k,p,h}$. We introduce the Sobolev space $W_p^k(\Omega)$, the closure of $C^\infty(\Omega)$ in the norm $\|\cdot\|_{k,p}$, and the Sobolev space $W_1^k(\Gamma)$, the closure of $C^\infty(\Gamma)$ in the norm

$$
\|v\|_{k,1,\Gamma} = \sum_{j \leqslant k} \int_\Gamma \left|\frac{d^j v}{ds^j}\right| ds,
$$

where d/ds denotes differentiation with respect to arc length. If $k = 0$, we omit this index. For example, $\|\cdot\|_{p,h}$ will thus denote the L_p-norm over Ω_h.

We can now state our convergence results.

THEOREM 1. *Let $u \in W_2^2(\Omega) \cap W_\infty^1(\Omega)$. Then, there is a constant C such that for* $0 < h < 1$,

$$
|u - u_h|_{1,2,h} \leqslant Ch.
$$

THEOREM 2. *Let $u \in W_q^2(\Omega)$ for some $q > 2$ and $\varphi \in W_1^2(\Gamma)$. Then, for any p with $1 \leqslant p < 2$, there is a constant C such that, for* $0 < h < 1$,

$$
\|u - u_h\|_{p,h} \leqslant Ch^2.
$$

The proofs of these estimates are given in Sections 2 and 3, respectively. For linear equations, such results are well known (cf., e.g., [3]); the latter then holds for $p = q = 2$.

2. Proof of Theorem 1. Since u_h minimizes the functional F over S_h, we find, taking first variations, denoting by $v_{,i}$ the derivative of v with respect to the ith variable, that

$$\int_{\Omega_h} f_{,i}(\nabla u_h)\chi_{,i}\,dx = \int_{\Omega_h} \frac{\nabla u_h \nabla \chi}{\sqrt{1 + |\nabla u_h|^2}}\,dx = 0 \quad \text{for } \chi \in \mathring{S}_h, \tag{2.1}$$

where $\mathring{S}_h$ is the set of continuous functions defined on Ω_h which are linear on each T_j and vanish on the boundary of Ω_h. Let us extend the functions in $\mathring{S}_h$ to be zero outside Ω_h. Then the functions in $\mathring{S}_h$ are Lipschitz continuous and vanish on the boundary of Ω so that, taking first variations in the continuous problem,

$$\int_{\Omega} f_{,i}(\nabla u)\chi_{,i}\,dx = \int_{\Omega_h} \frac{\nabla u \nabla \chi}{\sqrt{1 + |\nabla u|^2}}\,dx = 0 \quad \text{for } \chi \in \mathring{S}_h. \tag{2.2}$$

Theorem 1 will be an obvious consequence of Lemmas 1 and 2 below.

LEMMA 1. *Let $u \in W_2^2(\Omega) \cap W_\infty^1(\Omega)$. Then, there is a constant C such that for* $0 < h < 1$,

$$\left(\int_{\Omega_h} \frac{|\nabla u - \nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}}\,dx\right)^{1/2} \leqslant Ch.$$

Proof. Let w_h be any function in S_h, and set $\chi = w_h - u_h$. Then $\chi \in \mathring{S}_h$ and, using (2.1) and (2.2), we find

$$\begin{aligned}
A^2 &= \int_{\Omega_h} \frac{|\nabla u - \nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}}\,dx \\
&= \int_{\Omega_h} \frac{(\nabla u - \nabla u_h)\nabla \chi}{\sqrt{1 + |\nabla u_h|^2}}\,dx + \int_{\Omega_h} \frac{(\nabla u - \nabla u_h)(\nabla u - \nabla w_h)}{\sqrt{1 + |\nabla u_h|^2}}\,dx \\
&= \int_{\Omega_h} \nabla u \nabla \chi \left(\frac{1}{\sqrt{1 + |\nabla u_h|^2}} - \frac{1}{\sqrt{1 + |\nabla u|^2}}\right) dx + \int_{\Omega_h} \frac{(\nabla u - \nabla u_h)(\nabla u - \nabla w_h)}{\sqrt{1 + |\nabla u_h|^2}}\,dx \\
&= D_1 + D_2.
\end{aligned}$$

For the second term, we find by Cauchy's inequality, $|D_2| \leqslant A|u - w_h|_{1,2,h}$. For the first term, we obtain with $\gamma = \max_{\overline{\Omega}} |\nabla u|/\sqrt{1 + |\nabla u|^2}$,

$$\begin{aligned}
|D_1| &\leqslant \int_{\Omega_h} |\nabla u|\,|\nabla \chi| \frac{|\nabla u - \nabla u_h|(|\nabla u| + |\nabla u_h|)}{\sqrt{1 + |\nabla u|^2}\sqrt{1 + |\nabla u_h|^2}\,(\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla u_h|^2})}\,dx \\
&\leqslant \gamma \int_{\Omega_h} \frac{|\nabla \chi|\,|\nabla u - \nabla u_h|}{\sqrt{1 + |\nabla u_h|^2}}\,dx \leqslant \gamma A \left(\int_{\Omega_h} \frac{|\nabla \chi|^2}{\sqrt{1 + |\nabla u_h|^2}}\,dx\right)^{1/2} \\
&\leqslant \gamma A(A + |u - w_h|_{1,2,h}).
\end{aligned}$$

Thus

$$A^2 \leqslant \gamma A(A + |u - w_h|_{1,2,h}) + A|u - w_h|_{1,2,h},$$

so that, since $\gamma < 1$,

$$A \leqslant (1 + \gamma)|u - w_h|_{1,2,h}/(1 - \gamma).$$

Now let w_h agree with u at the nodes. By a well-known estimate (cf., e.g., [3]), we then have

$$|u - w_h|_{1,2,h} \leqslant Ch|u|_{2,2},$$

which completes the proof of the lemma.

As a consequence of Lemma 1, we find

$$\|\nabla u - \nabla u_h\|_{1,h} \leqslant \left(\int_{\Omega_h} \frac{|\nabla u - \nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}}\right)^{1/2} \left(\int_{\Omega_h} \sqrt{1 + |\nabla u_h|^2}\, dx\right)^{1/2} \leqslant Ch, \tag{2.3}$$

since, clearly, $\int_{\Omega_h} \sqrt{1 + |\nabla u_h|^2}\, dx$ is bounded as a result of the minimizing property of u_h. In fact, Lemma 1 and (2.3) hold without the assumption that the edges of the triangles have length bounded below by ch. This assumption, however, will enter in the proof of the following lemma.

LEMMA 2. *Let $u \in W_2^2(\Omega) \cap W_\infty^1(\Omega)$. Then, there is a constant C such that for any $0 < h < 1$, $\|\nabla u_h\|_{\infty,h} \leqslant C$.*

Proof. By Lemma 1, we have, in particular, for any $T_j \subset \overline{\Omega}_h$,

$$\left(\int_{T_j} \frac{|\nabla u - \nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}}\, dx\right)^{1/2} \leqslant Ch,$$

so that

$$\left(\int_{T_j} \frac{|\nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}}\, dx\right)^{1/2} \leqslant Ch + C|u|_{1,\infty}\left(\int_{T_j} dx\right)^{1/2} \leqslant Ch.$$

Since ∇u_h is constant on T_j, and the area of T_j is bounded from below by a constant times h^2, it follows that

$$\frac{|\nabla u_h|^2}{\sqrt{1 + |\nabla u_h|^2}} \leqslant C \quad \text{on } T_j,$$

and thus $\max_{\overline{\Omega}_h} |\nabla u_h| \leqslant C$, which proves the lemma.

Together with Lemma 1, this also completes the proof of Theorem 1.

3. Proof of Theorem 2. We shall now prove Theorem 2 using an adaptation of a duality argument employed previously for linear problems by, e.g., Nitsche [3].

For technical reasons, we shall need to extend u_h to a piecewise linear function defined on the polygonal domain $\widetilde{\Omega}_h \supset \Omega$ consisting of the union of the triangles which intersect $\overline{\Omega}$. To this end, we first extend $u \in W_q^2(\Omega)$ to a domain $\widetilde{\Omega}$ with $\widetilde{\Omega} \supset \Omega_h$ for $0 < h < 1$ in such a way that the extended u belongs to $W_q^2(\widetilde{\Omega})$ (cf. [1]). We then extend u_h to $\widetilde{\Omega}_h$ by setting u_h equal to the linear function which interpolates the extended u at the vertices of T_j for each $T_j \subset \widetilde{\Omega}_h \backslash \Omega_h$. It is clear that, with u_h extended in this fashion, the estimate of Theorem 1 holds, with Ω_h replaced by Ω, i.e., $|u - u_h|_{1,2} \leqslant Ch$.

We shall prove that, for any p with $1 \leqslant p < 2$, there is a constant C such that $\|u - u_h\|_p \leqslant Ch^2$, which implies Theorem 2 since $\Omega \supset \Omega_h$. By increasing p or decreasing q, we may assume without loss of generality that $1/p + 1/q = 1$. It will, therefore, be sufficient to prove that there is a constant C such that

$$|(g, u - u_h)| = \left| \int_\Omega g(u - u_h)\, dx \right| \leqslant Ch^2 \|g\|_q \quad \text{for } g \in L_q(\Omega). \tag{3.1}$$

This will be accomplished by rewriting the left-hand side, interpreting g as the right-hand side of a certain linear elliptic equation.

For this purpose, let us start with the simple identity

$$\int_\Omega [f_{,i}(\nabla u) - f_{,i}(\nabla u_h)]\, \chi_{,i}\, dx = \int_\Omega a_{ij}^h (u - u_h)_{,j} \chi_{,i}\, dx, \tag{3.2}$$

where, for $x \in \Omega$,

$$a_{ij}^h(x) = \int_0^1 f_{,ij}(\nabla u_h(x) + s(\nabla u(x) - \nabla u_h(x)))\, ds, \qquad i, j = 1, 2.$$

Defining the bilinear form

$$a_h(\chi, \psi) = \int_\Omega a_{ij}^h \chi_{,i} \psi_{,j}\, dx,$$

we notice that, by (2.1), (2.2) and (3.2), we have

$$a_h(\chi, u - u_h) = 0 \quad \text{for } \chi \in \mathring{S}_h. \tag{3.3}$$

Since the coefficients of a_h are discontinuous, it will be convenient to introduce also the bilinear form

$$a(\chi, \psi) = \int_\Omega a_{ij} \chi_{,i} \psi_{,j}\, dx \quad \text{with } a_{ij}(x) = f_{,ij}(\nabla u(x)).$$

Since $u \in W_q^2(\Omega)$ and, in particular, ∇u is bounded, we find, using also (1.1), that the coefficients a_{ij} satisfy the assumptions in the following lemma:

LEMMA 3. *Assume that $a_{ij} \in W_q^1(\Omega)$ for some $q > 2$ and that $a_{ij}(x)\xi_i\xi_j$ is uniformly elliptic in Ω. Then, there exists a constant C such that, for any $g \in L_q(\Omega)$, the Dirichlet problem*

$$-(a_{ij} v_{,i})_{,j} = g \quad \text{in } \Omega, \quad v = 0 \text{ on } \Gamma, \tag{3.4}$$

admits a unique solution $v \in W_q^2(\Omega)$ and

$$\|v\|_{2,q} \leqslant C\|g\|_q. \tag{3.5}$$

Proof. See [4, p. 203].

Multiplying (3.4) by $u - u_h$ and integrating by parts, we now find that $(g, u - u_h)$ can be rewritten in the following way:

$$(g, u - u_h) = a(v, u - u_h) + \int_\Gamma v_n(u - u_h)\,ds$$

$$= a_h(v, u - u_h) + (a - a_h)(v, u - u_h) + \int_\Gamma v_n(u - u_h)\,ds.$$

Here $v_n = -n_j a_{ij} v_{,i}$, where (n_1, n_2) is the outward normal to Γ. We shall prove that each of the three last terms is bounded by $Ch^2 \|g\|_q$, which will obviously prove the desired inequality (3.1).

To estimate the first term, let $v_h \in \mathring{S}_h$ interpolate v on Ω, so that $|v - v_h|_{1,2} \leqslant Ch|v|_{2,2}$. Since the coefficients of a_h are bounded (cf. (1.1)), we thus find, by (3.3), (3.5) and Theorem 1, that

$$|a_h(v, u - u_h)| = |a_h(v - v_h, u - u_h)| \leqslant C|v - v_h|_{1,2}|u - u_h|_{1,2}$$

$$\leqslant Ch^2|v|_{2,2} \leqslant Ch^2\|g\|_2 \leqslant Ch^2\|g\|_q.$$

Consider next the second term $(a - a_h)(v, u - u_h)$. Since the derivatives of the $f_{,ij}$ are bounded in R^2, we have

$$|a_{ij} - a_{ij}^h| = \int_0^1 [f_{,ij}(\nabla u) - f_{,ij}(\nabla u_h + s\nabla(u - u_h))]\,ds$$

$$\leqslant C|\nabla u - \nabla u_h| \quad \text{in } \Omega,$$

so that

$$\|a_{ij} - a_{ij}^h\|_2 \leqslant C|u - u_h|_{1,2}, \qquad i, j = 1, 2.$$

Further, by Sobolev's inequality and Lemma 3,

$$|v|_{1,\infty} \leqslant C|v|_{2,q} \leqslant C\|g\|_q. \tag{3.6}$$

Thus by Theorem 1,

$$|(a - a_h)(v, u - u_h)| \leqslant C|v|_{1,\infty} \max_{i,j} \|a_{ij} - a_{ij}^h\|_2 |u - u_h|_{1,2}$$

$$\leqslant Ch^2\|g\|_q.$$

Finally, for the boundary term, we have by (3.6)

$$\left|\int_\Gamma v_n(\varphi - u_h)\,ds\right| \leqslant C|v|_{1,\infty}\|\varphi - u_h\|_{1,\Gamma} \leqslant C\|g\|_q\,\|\varphi - u_h\|_{1,\Gamma}.$$

It is therefore sufficient to prove that

$$\|\varphi - u_h\|_{1,\Gamma} \leqslant Ch^2. \tag{3.7}$$

To see this, let φ_h be the piecewise linear function of arc length s defined on Γ which agrees with φ at the vertices on Γ. We then clearly have that $\|\varphi - \varphi_h\|_{1,\Gamma} \leqslant Ch^2 |\varphi|_{2,1,\Gamma}$, and therefore (3.7) will follow if we prove that $\|\varphi_h - u_h\|_{1,\Gamma} \leqslant Ch^2$. To show this, we argue as follows: For any $\bar{P} \in \Gamma$, let T_j be the triangle in $\widetilde{\Omega}_h \backslash \Omega_h$ such that $\bar{P} \in T_j$. Let P_1 and P_2 be the vertices of T_j on Γ, let s_1 and s_2 be the arc lengths corresponding to P_1 and P_2, and assume that $\bar{P}$ corresponds to $s = s_1 + \lambda(s_2 - s_1)$ where $0 \leqslant \lambda \leqslant 1$. Let now P be the point on the chord P_1P_2 such that $\operatorname{dist}(P, P_1) = \lambda \operatorname{dist}(P_1, P_2)$. Since we are interpolating linearly, we then have $\varphi_h(\bar{P}) = u_h(P)$. It is easy to see that $\operatorname{dist}(\bar{P}, P) \leqslant Ch^2$. Further, since u_h is the interpolant of u on T_j, we have that $|\nabla u_h|$ is bounded on T_j and therefore

$$|\varphi_h(\bar{P}) - u_h(\bar{P})| = |u_h(P) - u_h(\bar{P})| \leqslant Ch^2 \quad \text{for } \bar{P} \in \Gamma,$$

which implies that $\|\varphi_h - u_h\|_{1,\Gamma} \leqslant Ch^2$. This completes the proof of Theorem 2.

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