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A Lower Bound on the Angles of Triangles Constructed by Bisecting the Longest Side

By Ivo G. Rosenberg and Frank Stenger

Abstract. Let $\Delta A^1A^2A^3$ be a triangle with vertices at A^1, A^2 and A^3. The process of "bisecting $\Delta A^1A^2A^3$" is defined as follows. We first locate the longest edge, A^iA^{i+1} of $\Delta A^1A^2A^3$ where $A^{i+3} = A^i$, set $D = (A^i + A^{i+1})/2$, and then define two new triangles, ΔA^iDA^{i+2} and $\Delta DA^{i+1}A^{i+2}$.

Let Δ_{00} be a given triangle, with smallest interior angle $\alpha > 0$. Bisect Δ_{00} into two new triangles, Δ_{1i}, $i = 1, 2$. Next, bisect each triangle Δ_{1i}, to form four new triangles Δ_{2i}, $i = 1, 2, 3, 4$, and so on, to form an infinite sequence T of triangles. It is shown that if $\Delta \in T$, and θ is any interior angle of Δ, then $\theta \geqslant \alpha/2$.

Results. Let ΔABC be a triangle with vertices at A, B and C. The procedure "bisect ΔABC" is defined as follows. We form two triangles from ΔABC by locating the midpoint of the longest side of ΔABC and drawing a straight line segment from this midpoint to the vertex of ΔABC which is opposite the longest side. (If there is more than one side of greatest length, we bisect any one of them.) For example, if BC is the longest side of ΔABC, we set $D = (B + C)/2$ to form two new triangles ΔABD and ΔADC.

Let ΔABC be a given triangle with interior angles α, β and γ located at A, B and C, respectively. We form an infinite family $T(A, B, C)$ of triangles as follows. We first bisect $\Delta_{00} \equiv \Delta ABC$ to form two new triangles Δ_{1i}, $i = 1, 2$. We next bisect each of these two triangles to form four new triangles Δ_{2i}, $i = 1, 2, 3, 4$. Next, we bisect each of these four triangles to form eight new triangles Δ_{3i}, $i = 1, 2, 3, \ldots, 8$, and so on.

It is convenient to apply this procedure of bisections in order to refine the mesh in the finite element approximations of solutions of differential equations (see, e.g., [1]). Recently [2], this procedure of bisecting triangles was used to obtain a two-dimensional analogue of the one-dimensional method of bisections for solving nonlinear equations. A criterion of convergence of the above procedures is that the interior angles of Δ_{ni} do not go to zero as $n \longrightarrow \infty$. The Schwarz paradox [3, pp. 373–374] provides an explicit example of a situation in which triangles are used to approximate the area of a cylinder. In this case, the sum of the areas may not converge to the area of the cylinder as the length of each side of the triangles approaches zero, and the number of triangles approaches infinity, if the smallest interior angle of each triangle approaches zero.

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In this note, we prove the following theorem, which ensures that the interior angles of Δ_{ni} do not go to zero as $n \longrightarrow \infty$.

THEOREM. *Let the smallest interior angle of ΔABC be λ, and let $0 < x_\lambda < \pi/4$ be the solution of*

$$\tan x_\lambda = \frac{\sin \lambda}{2 - \cos \lambda}. \tag{1}$$

If Δ is a triangle in $T(A, B, C)$, and θ is an interior angle of Δ, then $\theta \geqslant x_\lambda$.

COROLLARY. *If $\Delta \in T(A, B, C)$ and θ is an interior angle of Δ, then $\theta \geqslant \lambda/2$.*

In the case when λ is small, x_λ is a better lower bound than $\lambda/2$, since $x_\lambda/\lambda \longrightarrow 1$ as $\lambda \longrightarrow 0$. For example, when $\lambda = \pi/6$, $x_\lambda \cong .777(\pi/6) > .5(\pi/6) = \lambda/2$.

Before we start the proof of the above theorem and corollary, we introduce the following notation.

Let ΔRST be a triangle with interior angles ρ, σ and τ at R, S and T, respectively. If ΔRST is bisected into two triangles $\Delta R_iS_iT_i$ with interior angles ρ_i, σ_i and τ_i located at R_i, S_i and T_i, respectively, $i = 1, 2$, we use both the notations

$$(\rho, \sigma, \tau) \longrightarrow (\rho_i, \sigma_i, \tau_i), \qquad (\rho_i, \sigma_i, \tau_i) \longleftarrow (\rho, \sigma, \tau). \tag{2}$$

As the notation suggests, (ρ, σ, τ) actually denotes a similarity class in $T(A, B, C)$ and "$\longrightarrow$" is a binary relation, or graph,on the set of all these similarity classes. We also use the notation $|M - N|$ to denote the Euclidean distance between the points M and N.

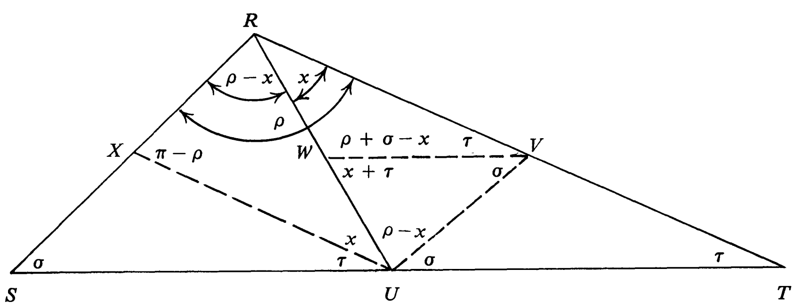

FIGURE 1. *Bisections of a triangle ΔRST*

Proof of the Theorem. Let ΔRST (see Fig. 1) belong to the family $\Delta(A, B, C)$, and let ΔRST have interior angles ρ at R, σ at S, and τ at T. Let us also assume without loss of generality that $0 < \tau \leqslant \sigma \leqslant \rho$. Since also $\rho + \sigma + \tau = \pi$, it follows that

$$\tau \leqslant \pi/3 \leqslant \rho < \pi \quad \text{and} \quad \sigma < \pi/2. \tag{3}$$

From Fig. 1, we obtain

$$(4) \qquad (x, \tau, \rho + \sigma - x) \longleftarrow (\rho, \sigma, \tau) \longrightarrow (\rho - x, \sigma, x + \tau).$$

Since the sizes of the sides of ΔRST are in the same relation as the opposite angles, from $\sigma \geqslant \tau$ we first get $|T - R| \geqslant |S - R|$, and then, applying the same principle to ΔRVU, the relations $|V - R| = \frac{1}{2}|T - R| \geqslant \frac{1}{2}|S - R| = |V - U|$ yield

$$(5) \qquad x \leqslant \rho - x.$$

LEMMA 1. *Let $\tau \leqslant \pi/3$ and $\rho = \sigma = \pi/2 - \tau/2$. Then the angle x_τ in Fig. 2 satisfies*

$$(6) \qquad \tan x_\tau = \frac{\sin \tau}{2 - \cos \tau} \geqslant \tan \tau/2.$$

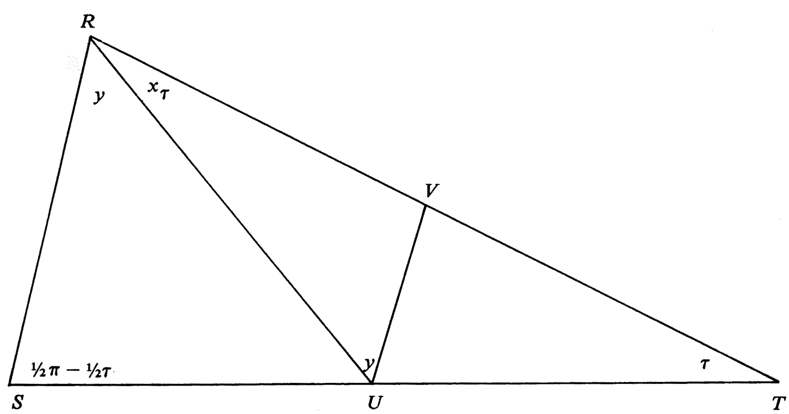

FIGURE 2. *ΔRST when $|R - T| = |S - T|$*

Proof. The law of sines in ΔRUV and ΔRST yields

$$\frac{\sin x_\tau}{\sin y} = \frac{|V - U|}{|V - R|} = \frac{|S - R|}{|T - R|} = \frac{\sin \tau}{\sin(\pi/2 - \tau/2)}.$$

Since $y = \pi/2 - \tau/2 - x_\tau$, we obtain

$$\sin x_\tau \cos \tfrac{1}{2}\tau = \sin \tau \cos(\tau/2 + x_\tau).$$

Simplifying, we get $\tan x_\tau = \sin \tau/(2 - \cos \tau)$. From the relation $2z \equiv \tau \leqslant \pi/3$, we get $\cos^2 z - \sin^2 z = \cos 2z \geqslant \frac{1}{2}$, and hence $2 \cos^2 z \geqslant 1 + 2 \sin^2 z = 2 - \cos 2z$. This yields $2 \sin z \cos z/(2 - \cos 2z) \geqslant \tan z$, which proves Lemma 1.

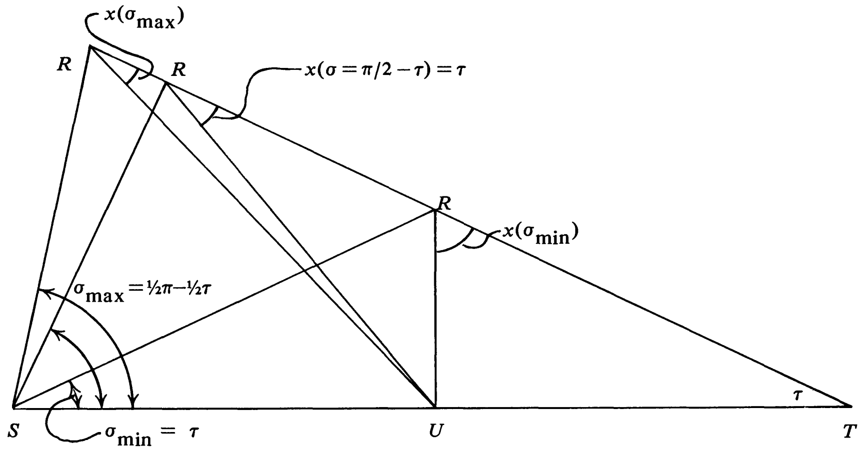

FIGURE 3. *Various values of the angle* x_τ

With reference to Fig. 3, let us fix S, T, and the angle τ, and change σ so that $\rho \geqslant \sigma \geqslant \tau$. Clearly, σ changes from $\sigma_{min} = \tau$ to $\sigma_{max} = \pi/2 - \tau/2$, when ΔRST becomes an isosceles triangle. Clearly, $x = x(\sigma)$ is a decreasing function of σ in the region $\sigma_{min} \leqslant \sigma \leqslant \sigma_{max}$, whose range of values are

$$x_\tau = x(\sigma_{max}) \leqslant x(\sigma) \leqslant x(\sigma_{min}) = \pi/2 - \tau,$$

where x_τ is defined in (6). Thus by Lemma 1,

$$x \geqslant x_\tau \geqslant \tau/2. \tag{7}$$

Notice also that when $\rho = \pi/2$, $\sigma = \pi/2 - \tau$, and $x(\sigma) = x(\pi/2 - \tau) = \tau$. It is thus evident from Fig. 3, that

$$x \geqslant \tau \Longleftrightarrow \rho \geqslant \pi/2. \tag{8}$$

Finally, we remark that x_τ is an increasing function of τ in the region $0 \leqslant \tau \leqslant \pi/3$, which can be easily verified by computing the derivative of x_τ using (6).

We next show that

$$x + \tau \leqslant \pi/2, \quad \rho + \sigma - x \geqslant \pi/2. \tag{9}$$

For if $x + \tau > \pi/2$, then, since the interior angles of ΔRSU in Fig. 1 add up to π, it would follow that $\rho + \sigma - x < \pi/2$. However, from (5), we get $\rho - x \geqslant \rho/2$, and so $\sigma + \rho/2 < \pi/2$, i.e., $\rho + 2\sigma < \pi$. Since, however, $\rho + 2\sigma \geqslant \rho + \sigma + \tau = \pi$, we arrive at a contradiction, i.e., (9) is valid.

In view of (9), we establish

LEMMA 2. *The following situation*

$$(10)\qquad \begin{array}{ccc} (\rho, \sigma, \tau) & \rightleftarrows & (x, \tau, \rho + \sigma - x) \\ \downarrow & & \downarrow \\ (\rho - x, \sigma, x + \tau) & & (x, \rho - x, \pi - \rho) \end{array}$$

is valid in general.

LEMMA 3. *If*

$$(11)\qquad \pi - \rho \geqslant \rho - x,$$

then

$$(12)\qquad \begin{array}{ccc} (\rho, \sigma, \tau) & \rightleftarrows & (x, \tau, \rho + \sigma - x) \\ \downarrow & & \uparrow\ \downarrow \\ (\rho - x, \sigma, x + \tau) & \longleftarrow & (x, \rho - x, \pi - \rho) \end{array}$$

Proof of Lemma 3. By combining (11) and (5), it follows that $\pi - \rho \geqslant \rho - x \geqslant x$, and (12) now follows by inspection of ΔRXU or ΔRUV.

We next consider the bisection of ΔWUV or ΔRSU.

LEMMA 4. *Let* (11) *hold. If*

$$(13)\qquad x + \tau \geqslant \sigma \quad \text{and} \quad x + \tau \geqslant \rho - x$$

or else if

$$(14)\qquad \rho - x < \tau,$$

then

$$(15)\qquad \begin{array}{ccc} (\rho, \sigma, \tau) & \rightleftarrows & (x, \tau, \rho + \tau - x) \\ \downarrow\ \uparrow & & \uparrow\ \downarrow \\ (\rho - x, \sigma, x + \tau) & \rightleftarrows & (x, \rho - x, \pi - \rho) \end{array}$$

Proof. If (13) is satisfied, then (15) clearly follows from (12) and inspection of ΔWUV in Fig. 1. If (14) holds, then $\rho - x < \tau < x + \tau$, so that the second relation in (13) is satisfied. If the first relation of (13) were not satisfied, then $\sigma > x + \tau$, and, by (14), $x + \tau > x + \rho - x = \rho$, i.e., $\sigma > \rho$, which contradicts our original assumption, that $\tau \leqslant \sigma \leqslant \rho$. This proves Lemma 4.

Let us now complete the proof of the theorem. Let us set $\nu = \nu(\rho, \sigma, \tau) = \min(\rho, \sigma, \tau)$. We shall show that, along the transition $\longrightarrow$, either (i) ν is nondecreasing, or (ii) we get four triples $t_i = (\rho_i, \sigma_i, \tau_i)$ such that $\nu = \nu(t_i) \geqslant x_T$, $i = 1, 2, 3, 4$, and such that if an arrow emanates from one of the four triples, t_j, to a triple t where

$t \neq t_i$, $i = 1, 2, 3, 4$, then $\nu(t_j) \geqslant \tau$. Since x_τ is an increasing function of τ, it will therefore be impossible to get $\nu(\Delta_{ni}) < x_\lambda$ for any $\Delta_{ni} \in T(A, B, C)$.

Let us assume that ΔRST is an arbitrary triangle in the family $T(A, B, C)$, such that $\rho \geqslant \sigma \geqslant \tau$. In (10), $\rho + \sigma - x \geqslant \pi/2 > \tau$, $\sigma \geqslant \tau$, $x + \tau \geqslant \tau$, and $\pi - \rho = \sigma + \tau \geqslant \tau$. Thus, the only candidates for angles $< \tau$ are x and $\rho - x$. If $\rho \geqslant \pi/2$, it follows from (5) and (8) that $\rho - x \geqslant x \geqslant \tau$, and hence $\nu(\rho - x, \sigma, x + \tau) \geqslant \nu(x, \tau, \rho + \sigma -$. $\geqslant \nu(\rho, \sigma, \tau)$.

Let us assume, therefore, that $\rho < \pi/2$. Then $\pi - \rho > \pi/2 > \rho > \rho - x$, so that (11) is satisfied, and, by Lemma 3, we get the situation (12). In (12), the inequalities $\sigma \geqslant \tau$, $\rho + \sigma - x \geqslant \pi/2$, and $\pi - \rho \geqslant \pi/2$ are valid, and so only the angles x and $\rho - x$ can be less than τ. By (7) and (5) $\rho - x \geqslant x \geqslant x_\tau$. The configuration (12) is such that arrows going outside of it can originate only at $(\rho - x, \sigma, x + \tau)$. If $\rho - x \geqslant \tau$, then $\nu(\rho - x, \sigma, x + \tau) \geqslant \tau$.

Finally, if $\rho - x < \tau$, we have the situation of Lemma 4, that is the four triples $(\rho_i, \sigma_i, \tau_i)$ form a "trap" in the sense that there are no arrows emanating from them, and such that $\nu(\rho_i, \sigma_i, \tau_i) \geqslant x_\tau$, $i = 1, 2, 3, 4$.

This clearly completes the proof.

Centre de Recherches Mathématiques
Université de Montréal
Montréal, Québec, Canada

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

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