MATHEMATICS OF COMPUTATION , VOLUME 29, NUMBER 130
APRIL 1975, PAGES 559–565

Some Properties of Orthogonal Polynomials

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Abstract. Some results are obtained concerning the signs of the coefficients in the expansions in powers of x^{-1}, $(1 + x)^{-1}$ or $(1 - x)^{-1}$ of $1/p_n(x)$ and $q_n(x)$, where $p_n(x)$ is the polynomial of degree n in the orthogonal sequence associated with a given weight-function $w(x)$ over $(-1, 1)$ and $q_n(x) = \int_{-1}^{1} w(t)p_n(t)(x - t)^{-1}\,dt$.

1. **Origin of the Problem.** The problem to be considered here has its origin in some results obtained by Stenger [5]. Let a weight-function $w(x)$ be positive and continuous in the interval $-1 < x < 1$, and such that $\int_{-1}^{1} w(x)\,dx$ exists. Then it is well known that there is a sequence of polynomials $\{p_0(x), p_1(x), \cdots\}$, $p_n(x)$ being of exact degree n, satisfying the orthogonality-relation

$$\int_{-1}^{1} w(x)p_m(x)p_n(x)\,dx = 0 \qquad (m \neq n) \tag{1}$$

(see, e.g., Szegö [7]). Each polynomial in the sequence is unique apart from a constant factor. We shall impose no particular normalisation on the polynomials, but shall merely stipulate that the coefficient of x^n in $p_n(x)$ is positive.

A second sequence of functions $\{q_0(x), q_1(x), \cdots\}$ can be defined in terms of the above orthogonal sequence by the equation

$$q_n(x) = \int_{-1}^{1} \frac{w(t)p_n(t)\,dt}{x - t}; \tag{2}$$

$q_n(x)$ is then analytic and single-valued in the complex plane cut along the interval $[-1, 1]$.

The two functions $p_n(x)$ and $q_n(x)$ have been widely used in recent years in analysing the error in the Gaussian quadrature formulae for integrals of the form $\int_{-1}^{1} w(x)f(x)\,dx$; among many references, we may mention Barrett [1], Donaldson and Elliott [2], Stenger [5]. Stenger's analysis is concerned largely with the signs of the coefficients $b_{n,j}$ and $c_{n,j}$ in the following two series, which both converge absolutely and uniformly for $|x| \geqslant R > 1$:

$$1/p_n(x) = \sum_{j=0}^{\infty} b_{n,j}x^{-n-j} \qquad (n \geqslant 1), \tag{3}$$

Received August 6, 1973.

AMS (MOS) subject classifications (1970). Primary 42A52; Secondary 41A55, 33A65.

Key words and phrases. Orthogonal polynomials, weight-function, power-series, Jacobi polynomials, Gaussian quadrature.

and

$$q_n(x) = \sum_{j=0}^{\infty} c_{n,j} x^{-n-j-1} \qquad (n \geqslant 0). \tag{4}$$

In particular, he shows that *if $w(x)$ is an even function of x*, then

(a) $b_{n,2j} > 0$ and $b_{n,2j+1} = 0$, except in the case $n = 1$, when $b_{1,0} > 0$ and $b_{1,j} = 0$ for $j > 0$;

(b) $c_{n,2j} > 0$ and $c_{n,2j+1} = 0$.

These results are, in fact, quite easily proved. The problem of determining the signs of the coefficients $b_{n,j}$ and $c_{n,j}$ when $w(x)$ is not an even function appears to be considerably more difficult. In Section 2, we prove two theorems which go part of the way towards solving the problem.

The functions $1/p_n(x)$ and $q_n(x)$ can also be expanded in negative powers of $(1 + x)$ or $(1 - x)$. The corresponding problem for those expansions can be completely solved, and the results are given in Section 3. Section 4 deals briefly with the important special case $w(x) = (1 - x)^\alpha(1 + x)^\beta$, $(\alpha, \beta > -1)$, associated with the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Finally, in Section 5, a number of further results are conjectured.

2. Expansions in Negative Powers of x.

THEOREM 1. *If $w(x)/w(-x)$ is strictly increasing for $-1 < x < 1$, then $b_{n,j} > 0$, $n = 1, 2, \cdots,$ $j = 0, 1, 2, \cdots$.*

Proof. Let the zeros of $p_n(x)$ be $x_1, x_2, \cdots, x_n$. It is well known (see, e.g., Szegö [7, Theorem 3.3.1]) that they are real and distinct and lie in the open interval $(-1, 1)$. We shall arrange them in descending order, so that $x_1 > x_2 > \cdots > x_n$. Now if k_n denotes the coefficient of x^n in $p_n(x)$, so that $k_n > 0$, we have

$$1/p_n(x) = k_n^{-1} x^{-n} \prod_{k=1}^{n} (1 - x_k/x)^{-1} = k_n^{-1} \sum_{j=0}^{\infty} h_j x^{-n-j}, \tag{5}$$

where $h_j \equiv h_j(x_1, x_2, \cdots, x_n)$ denotes the homogeneous product sum of degree j of $x_1, x_2, \cdots, x_n$ (see, e.g., Littlewood [4, eq. 5.2]). Thus $b_{n,j} = h_j/k_n$, and it remains to show that $h_j > 0$.

Now let

$$w(x, \tau) = \tau w(x) + (1 - \tau) w(-x) \qquad (0 \leqslant \tau \leqslant 1) \tag{6}$$

so that, in particular, $w(x, 1) = w(x)$ and $w(x, 0) = w(-x)$. Further, let the zeros of the polynomial of degree n in the orthogonal sequence associated with weight-function $w(x, \tau)$ be $x_1(\tau) > x_2(\tau) > \cdots > x_n(\tau)$.

We have

$$\frac{w_\tau(x, \tau)}{w(x, \tau)} = \frac{w(x) - w(-x)}{\tau w(x) + (1 - \tau) w(-x)} = \tau^{-1} - \frac{\tau^{-1}}{w(x)/w(-x) - 1 + \tau^{-1}},$$

and this, under the conditions of the theorem, is a strictly increasing function of x. Hence, by a theorem of Markoff, (see Szegö [7, Theorem 6.12.1]), the kth zero $x_k(\tau)$ is an increasing function of τ. Now clearly, $w(x, \frac{1}{2})$ is an even function of x; consequently, $x_k(\frac{1}{2}) + x_{n-k+1}(\frac{1}{2}) = 0$.

Hence

$$x_k + x_{n-k+1} = x_k(1) + x_{n-k+1}(1) > 0.$$

It follows that for $r \geqslant 0$, $x_k^r + x_{n-k+1}^r > 0$.

Thus, if

$$S_r = \sum_{k=1}^{n} x_k^r, \tag{7}$$

then $S_r > 0$. But the functions h_j are expressible in terms of the S_r:

$$h_j = \sum_{(\alpha)} \frac{1}{\alpha_1!\alpha_2! \cdots \alpha_j!} \left(\frac{S_1}{1}\right)^{\alpha_1} \left(\frac{S_2}{2}\right)^{\alpha_2} \cdots \left(\frac{S_j}{j}\right)^{\alpha_j} \tag{8}$$

(see, e.g., Littlewood [4, p. 267]), the summation being over all partitions $(\alpha) = (1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j})$ of j. So, clearly, $h_j > 0$, proving the theorem.

COROLLARY. *If $w(x)/w(-x)$ is strictly decreasing for $-1 < x < 1$, then* $(-1)^j b_{n,j} > 0$, $n = 1, 2, \cdots$, $j = 0, 1, 2, \cdots$.

THEOREM 2. $c_{n,2j} > 0$, $n = 0, 1, 2, \cdots$, $j = 0, 1, 2, \cdots$.

Proof. By expanding $(x - t)^{-1}$ as a power-series in t/x, inserting in (2), and integrating term-by-term, we deduce that

$$c_{n,j} = \int_{-1}^{1} w(t) p_n(t) t^{n+j}\, dt. \tag{9}$$

Now, according to Hildebrand [3, Section 7.4], there is a function $U_n(x)$ with the following properties:

$$\begin{aligned} & w(x)p_n(x) = \frac{d^n}{dx^n} U_n(x) \qquad (-1 < x < 1), \\ & U_n(-1) = U_n'(-1) = \cdots = U_n^{(n-1)}(-1) = 0, \\ & U_n(1) = U_n'(1) = \cdots = U_n^{(n-1)}(1) = 0. \end{aligned} \tag{10}$$

Integrating (9) by parts n times and using (10), we thus obtain the result

$$c_{n,j} = \frac{(n+j)!}{j!} \int_{-1}^{1} (-1)^n U_n(x) x^j\, dx. \tag{11}$$

We now show that $(-1)^n U_n(x) > 0$ if $-1 < x < 1$. For $U_n^{(n)}(x)$ $(\equiv w(x)p_n(x))$ has n real zeros in $(-1, 1)$, i.e., $U_n^{(n-1)}(x)$ has n stationary points in the interval. It follows that $U_n^{(n-1)}(x)$ has at most $n+1$ zeros in the closed interval $[-1, 1]$. Since two of these are accounted for by the zeros at $x = \pm 1$, $U_n^{(n-1)}(x)$ has at most $n-1$ zeros in $(-1, 1)$. Similarly, $U_n^{(n-2)}(x)$ has at most $n-2$ zeros in $(-1, 1)$, and so on, until, eventually, we see that $U_n(x)$ is of constant sign in $(-1, 1)$. To establish the sign, we note that

$$\int_{-1}^{1} (-1)^n U_n(x)dx = c_{n,0}/n! = \frac{1}{n!}\int_{-1}^{1} w(x)p_n(x)x^n\,dx$$

$$= \frac{1}{n!k_n}\int_{-1}^{1} w(x)\{p_n(x)\}^2 dx > 0,$$

so that $(-1)^n U_n(x) > 0$. Thus, if j is even, the integrand in (11) is positive, and this completes our proof.

3. Expansions in Negative Powers of $(1+x)$ **or** $(1-x)$. The functions $1/p_n(x)$ and $q_n(x)$ can also be expanded in negative powers of $(1+x)$ or $(1-x)$, as follows:

$$(12)\qquad \begin{aligned} 1/p_n(x) &= \sum_{j=0}^{\infty} \beta_{n,j}(1+x)^{-n-j} \qquad (n \geqslant 1) \\ &= \sum_{j=0}^{\infty} \beta'_{n,j}(1-x)^{-n-j} \qquad (n \geqslant 1), \end{aligned}$$

$$(13)\qquad \begin{aligned} q_n(x) &= \sum_{j=0}^{\infty} \gamma_{n,j}(1+x)^{-n-j-1} \qquad (n \geqslant 0) \\ &= \sum_{j=0}^{\infty} \gamma'_{n,j}(1-x)^{-n-j-1} \qquad (n \geqslant 0). \end{aligned}$$

The two expansions in powers of $(1+x)^{-1}$ are absolutely and uniformly convergent if $|1+x| \geqslant R > 2$, those in powers of $(1-x)^{-1}$ if $|1-x| \geqslant R > 2$.

The problem of determining the signs of the coefficients $\beta_{n,j}, \beta'_{n,j}, \gamma_{n,j}$ and $\gamma'_{n,j}$ can be solved completely. The results are stated below, without proof, since the proofs are similar to those of Theorems 1 and 2 (and, in fact, rather simpler).

THEOREM 3.

$$(14)\qquad \left.\begin{aligned} \beta_{n,j} &> 0 \\ (-1)^n\beta'_{n,j} &> 0 \end{aligned}\right\} \quad n = 1, 2, \cdots, \; j = 0, 1, 2, \cdots.$$

THEOREM 4.

$$(15) \qquad \left.\begin{aligned} \gamma_{n,j} &> 0 \\ (-1)^{n+1}\gamma'_{n,j} &> 0 \end{aligned}\right\} \qquad n = 0, 1, 2, \cdots, \; j = 0, 1, 2, \cdots.$$

4. Application to the Jacobi Polynomials. Let

$$(16) \qquad w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \qquad (\alpha, \beta > -1).$$

Then, apart possibly from a scale-factor,

$$(17) \qquad p_n(x) = P_n^{(\alpha,\beta)}(x),$$

the Jacobi polynomial of degree n associated with $w(x)$. Also, $q_n(x)$ is closely related to the Jacobi function of the second kind, $Q_n^{(\alpha,\beta)}(x)$; in fact, from Szegö [7, Eq. (4.61.4)],

$$(18) \qquad q_n(x) = 2(x - 1)^{\alpha}(x + 1)^{\beta}Q_n^{(\alpha,\beta)}(x).$$

If $\alpha = \beta$, $w(x)$ is an even function, so that Stenger's results quoted in Section 1 apply. As for the other cases, if $\alpha < \beta$, then $w(x)/w(-x)$ is increasing, so that, by Theorem 1, $b_{n,j} > 0$, while if $\alpha > \beta$, $w(x)/w(-x)$ is decreasing, and the coefficients $b_{n,j}$ alternate in sign.

In fact, in this case, we can also determine the signs of the coefficients $c_{n,2j+1}$, since there is an explicit expression for the function $U_n(x)$ of Eq. (10), namely,

$$(19) \qquad U_n(x) = ((-1)^n/2^n n!)(1 - x)^{n+\alpha}(1 + x)^{n+\beta};$$

this corresponds to Rodrigues' formula, see Szegö [7, Eq. (4.3.1)]. So, from (11),

$$c_{n,2j+1} = \frac{(n + 2j + 1)!}{2^n n!(2j + 1)!} \int_{-1}^{1} x^{2j+1}(1 - x)^{n+\alpha}(1 + x)^{n+\beta}\,dx,$$

which is positive or negative according as $\alpha < \beta$ or $\alpha > \beta$.

Thus the signs of $b_{n,j}$ and $c_{n,j}$ are completely determined in this case. The expansion of $Q_n^{(\alpha,\beta)}(x)$ in powers of $(1 - x)^{-1}$ is given in Szegö [7, (4.61.5)].

5. Some Conjectures. The results proved in the last two sections suggest two further problems:

(a) Can we say anything about the sign of $b_{n,2j}$ for a general weight-function $w(x)$?

(b) Under what circumstances can we guarantee that $c_{n,2j+1} > 0$?

As to the first problem, it may be conjectured that $b_{n,2j} > 0$. This would follow immediately if the following purely algebraic conjecture could be proved:

CONJECTURE. *$h_{2j}(x_1, x_2, \cdots, x_n)$ is positive definite for real $x_1, x_2, \cdots, x_n$.*

Some related problems are dealt with in Szegö [6]. I have been able to prove the result only in the following special cases:

(i) $j = 1$ (all values of n).

(ii) $n \leqslant 3$ (all values of j).

(iii) $j = 2$, $n \leqslant 10$.

The proofs in these cases are outlined below.

(i) When $j = 1$, Eq. (8) becomes $h_2 = \frac{1}{2}(S_2 + S_1^2)$, which is clearly positive definite.

(ii) The result is obvious when $n = 1$. When $n = 2$, we have

$$h_{2j}(x_1, x_2) = (x_1^{2j+1} - x_2^{2j+1})/(x_1 - x_2),$$

and this is positive, since the numerator and denominator have the same sign.

The case $n = 3$ is more difficult. If the three variables have the same sign, h_{2j} is clearly positive. So we may assume that, say, $x_1 \geqslant x_2 > 0 > x_3$. Further, if $x_1 + x_3 \geqslant 0$, it follows from the final part of the proof of Theorem 1 that h_{2j} is positive. So there remains the case $|x_3| > |x_1|$.

From Littlewood [4, Chapter VI, Theorem V],

$$h_{2j}(x_1, x_2, x_3) = \begin{vmatrix} x_1^{2j+2} & x_1 & 1 \\ x_2^{2j+2} & x_2 & 1 \\ x_3^{2j+2} & x_3 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}$$

$$= \frac{(x_1 - x_3)(x_1^{2j+2} - x_2^{2j+2}) + (x_1 - x_2)(x_3^{2j+2} - x_1^{2j+2})}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$> 0.$$

(iii) The detailed argument in the case $j = 2$ is rather involved, and only a brief summary is given. When $j = 2$, Eq. (8) becomes

$$h_4 = \frac{1}{24}(6S_4 + 8S_3S_1 + 3S_2^2 + 6S_2S_1^2 + S_1^4).$$

The only term which can be negative is that involving S_3S_1. We shall now impose the constraint

(20) $$S_2 = 1.$$

The minimum value of $S_4(x_1, x_2, \cdots, x_n)$ subject to this constraint is $1/n$. Hence

$$h_4 \geqslant \frac{1}{24}(6/n + 8S_3S_1 + 3).$$

So, if we can show that the minimum value of $6/n + 8S_3S_1 + 3$ subject to (20) is positive, this proves the result. The problem can be tackled by using Lagrange multipliers.

Unfortunately, a separate argument is required for each value of n. It turns out that the required minimum is positive for $n \leqslant 10$; for example, when $n = 4$, there is a minimum value $6/4 + 8S_3S_1 + 3 = 3.677 \cdots$ when

$$x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}} \sin \frac{7\pi}{9}, \quad x_4 = \cos \frac{7\pi}{9},$$

and for 7 other sets of values of the variables, obtained by reversing the sign of all of them and/or permuting them. All the other stationary values when $n = 4$ correspond to positive values of S_3S_1.

The argument fails when $n = 11$; then there is a stationary point for which $6/11 + 8S_3S_1 + 3$ is negative.

In principle, it would be possible, of course, to minimise h_4 itself rather than $6/n + 8S_3S_1 + 3$, but the algebra then becomes very involved.

As to problem (b), the example of the Jacobi weight-function $w(x) = (1-x)^\alpha(1+x)^\beta$ suggests that $c_{n,2j+1} > 0$ if $w(x)/w(-x)$ is strictly increasing, but again no proof has been found.

6. Acknowledgement. I am grateful to the referee for pointing out to me the results contained in Section 4.

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