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A Class of Hessenberg Matrices with Known Pseudoinverse and Drazin Inverse

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Abstract. In this paper, a class of Hessenberg matrices is presented for adoption as test matrices. The Moore-Penrose inverse and the Drazin inverse for each member of this class are determined explicitly.

1. Introduction. Most numerical problems associated with solving a system of linear equations involve only rational numbers. However, square matrices over the real number field are considered in this paper.

Howell and Gregory [6] have shown how to avoid problems which arise in solving the matrix equation $Ax = b$ as a result of rounding errors in computer schemes. Specifically, they have shown how to use residue arithmetic to avoid ill-conditioned problems. Using a similar approach, Stallings and Boullion [12] have shown how to significantly reduce rounding errors in computer schemes which compute the Moore-Penrose inverse (pseudoinverse) for a given matrix. However, the rounding errors are not necessarily completely eliminated.

Chow [2] has presented a class of Hessenberg matrices which may be used as test matrices in checking the accuracy of matrix inversion programs. In this paper, a class of Hessenberg matrices is presented such that the pseudoinverse and Drazin inverse can be explicitly computed for each member. Furthermore, the eigenvalues and eigenvectors are known for the members of this class. Therefore, it appears reasonable that such a class of matrices may be useful as test matrices.

2. Definitions and Notation. One should distinguish between the class of matrices in [2] which are offered as test matrices and the class given below. Only square matrices over the real number field are considered.

Definition 2.1. The pseudoinverse of a matrix A is the unique solution A^+ of the four matrix equations $AXA = A$, $XAX = X$, $(AX)^T = AX$ and $(XA)^T = XA$ where $(\ )^T$ denotes the matrix transpose.

Definition 2.2. The index of a matrix A is the smallest nonnegative integer $\operatorname{Ind}(A) = k$ such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$.

Definition 2.3. The Drazin inverse of a matrix A is the unique solution A^D of the three matrix equations $AX = XA$, $XAX = X$, $A^{k+1}X = A^k$, where $\operatorname{Ind}(A) = k$.

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H_n shall denote the *Hessenberg matrix* of order n, where

$$H_n = \begin{bmatrix} \alpha & 1 & \cdot & \cdot & \cdot \\ \alpha^2 & \alpha & 1 & \cdot & \cdot \\ \alpha^3 & \alpha^2 & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \alpha^n & \alpha^{n-1} & \cdot & \cdot & \alpha \end{bmatrix},$$

and α is an arbitrary real number.

The eigenvalues of H_n were determined in [2]. H_n has $k = [n/2]$ eigenvalues equal to 0 (where $[n/2]$ denotes the largest integer not exceeding $n/2$) and whose remaining eigenvalues are

$$4\alpha \cos^2\left(\frac{m\pi}{n+2}\right), \qquad m = 1, 2, \ldots, n-k.$$

The reader can refer to [4] for the corresponding eigenvectors.

3. Pseudoinverse of H_n. There are several algorithms available for computing $(H_n)^+$, [1], [3], [5], [10]. The general form for $(H_n)^+$ is presented in this section.

Case 1 ($n = 2$). If H_2 is the matrix

$$H_2 = \begin{bmatrix} \alpha & 1 \\ \alpha^2 & \alpha \end{bmatrix}, \quad \text{then} \quad (H_2)^+ = \begin{bmatrix} \dfrac{\alpha}{(\alpha^2+1)^2} & \dfrac{\alpha^2}{(\alpha^2+1)^2} \\ \dfrac{1}{(\alpha^2+1)^2} & \dfrac{\alpha}{(\alpha^2+1)^2} \end{bmatrix}.$$

This can be easily verified by direct substitution into the four defining equations.

Case 2 ($n \geqslant 3$). If H_n is the Hessenberg matrix of order $n \geqslant 3$, then

$$(H_n)^+ = \begin{bmatrix} \dfrac{\alpha}{\alpha^2+1} & & & & & & \\ \dfrac{1}{\alpha^2+1} & 0 & & & & & \\ -\alpha & 1 & 0 & & & & \\ & -\alpha & 1 & 0 & & & \\ & & -\alpha & 1 & 0 & & \\ & & & \cdots & \cdots & \cdots & \\ & & & -\alpha & 1 & 0 & \\ & & & & -\alpha & \dfrac{1}{\alpha^2+1} & \dfrac{\alpha}{\alpha^2+1} \end{bmatrix}.$$

As in Case 1, this can be easily verified by direct substitution after noting

$$H_n(H_n)^+ = \left[\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{0} \\ \hline & \dfrac{1}{\alpha^2+1} & \dfrac{\alpha}{\alpha^2+1} \\ 0 & & \\ & \dfrac{\alpha}{\alpha^2+1} & \dfrac{\alpha^2}{\alpha^2+1} \end{array}\right], \quad (H_n)^+H_n = \left[\begin{array}{cc|c} \dfrac{\alpha^2}{\alpha^2+1} & \dfrac{\alpha}{\alpha^2+1} & \\ & & 0 \\ \dfrac{\alpha}{\alpha^2+1} & \dfrac{1}{\alpha^2+1} & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right]$$

where I_{n-2} is the identity matrix of order $n-2$.

4. **Drazin Inverse of H_n.** The index of H_n is first determined. If $\alpha = 0$, then H_n is nilpotent and $\operatorname{Ind}(H_n) = n$.

PROPOSITION. *If $\alpha \neq 0$, then* $\operatorname{Ind}(H_n) = [n/2]$.

Proof. Meyer [7] has shown that k is the index of H_n if k is the smallest integer such that $\operatorname{limit}_{\epsilon\to 0} \epsilon^k(H_n + \epsilon I_n)^{-1}$ exists. From [2] it is known that, if $(h_{ij}) = (H_n + \epsilon I_n)^{-1}$, then

$$h_{ij} = \frac{(-1)^{i+j}\Delta'_{i-1}\Delta_{n-j+1}}{\epsilon\Delta'_n}, \quad \text{if } i \leqslant j,$$

and

$$h_{ij} = \frac{-\alpha(\alpha\epsilon)^{i-j}\Delta'_{j-2}\Delta_{n-i}}{\Delta'_n}, \quad \text{if } i > j,$$

where

$$\Delta_0 = 1, \qquad \Delta'_0 = 1, \qquad \Delta'_{-1} = 1/\epsilon,$$

$$\Delta_1 = \epsilon, \qquad \Delta'_1 = \alpha + \epsilon,$$

$$\Delta_t = \epsilon\Delta_{t-1} + \alpha\epsilon\Delta_{t-2}, \qquad \Delta'_t = \Delta_t + \alpha\Delta_{t-1}.$$

Each Δ_i is a polynomial of degree i in ϵ.

First, in computing the

$$\operatorname*{limit}_{\epsilon\to 0} \epsilon^k h_{ij} = \operatorname*{limit}_{\epsilon\to 0} \frac{\epsilon^k(-1)^{i+j}\Delta'_{i-1}\Delta_{n-j+1}}{\epsilon\Delta'_n}$$

(when $i \leqslant j$) attention is directed to the terms of smallest power in ϵ of Δ_t and Δ'_t. Observe that

(1) The exponent of ϵ in the term of smallest degree in Δ_t is $t/2$ when t is even and $(t+1)/2$ otherwise,

(2) the exponent of ϵ in the term of smallest degree in Δ'_t is $[t/2]$.

Since, for a given n, $\epsilon\Delta'_n$ is fixed, the value of k depends on $\Delta'_{i-1}\Delta_{n-j+1}$. In this polynomial, the exponent of ϵ is minimum when i is smallest and j is largest. Therefore, the integer k for which $\operatorname{limit}_{\epsilon\to 0}\epsilon^k h_{1n}$ exists is also an integer for which $\operatorname{limit}_{\epsilon\to 0}\epsilon^k h_{ij}$ exists $(i \leqslant j)$. Now

$$\operatorname*{limit}_{\epsilon\to 0} \epsilon^k h_{1n} = \operatorname*{limit}_{\epsilon\to 0} \frac{\epsilon^k(-1)^{1+n}\Delta'_0\Delta_1}{\epsilon\Delta'_n} = \operatorname*{limit}_{\epsilon\to 0} \frac{(-1)^{1+n}\epsilon^{k+1}}{\epsilon\Delta'_n}$$

exists if $k \geqslant [n/2]$.

Second, when $i > j$, a similar argument will show that $\operatorname{limit}_{\epsilon\to 0} \epsilon^k h_{ij}$ exists whenever $\operatorname{limit}_{\epsilon\to 0} \epsilon^k h_{n1}$ exists, which is true when $k \geqslant [n/2]$.

Therefore, $\operatorname{limit}_{\epsilon\to 0} \epsilon^k (H_n + \epsilon I_n)^{-1}$ exists whenever $k \geqslant [n/2]$. To see that k is the smallest such integer observe that

$$\operatorname*{limit}_{\epsilon\to 0} \epsilon^{[n/2]-1} h_{1n} = \operatorname*{limit}_{\epsilon\to 0} \frac{\epsilon^{[n/2]-1}(-1)^{1+n}\epsilon}{\epsilon\Delta_n'}$$

does not exist since the exponent of ϵ in the term of smallest degree in Δ_n' is $[n/2]$. This completes the proof.

The Drazin inverse is now determined for H_n, using elementary divisor theory (see [8]) and a technique described in [9]. Consider the characteristic matrix $(H_n - \lambda I_n)$. Let $P_n(\lambda)$ denote the characteristic polynomial of H_n where $P_n(\lambda) = \det(H_n - \lambda I_n)$. Also,

$$P_n(\lambda) = (\alpha - \lambda)P_{n-1}(\lambda) + \sum_{j=1}^{n-1} (-1)^{n-j}\alpha^{n+1-j}P_{j-1}(\lambda),$$

where $P_0(\lambda) = 1$, $P_1(\lambda) = \alpha - \lambda$, and $P_n(\lambda) + \lambda P_{n-1}(\lambda) + \alpha\lambda P_{n-2}(\lambda) = 0$ (expanding by rows). The solution of this last equation [4] is

$$P_n(\lambda) = (-\alpha)^n 2^{n-1}(\lambda/4\alpha)^{(n-1)/2} \frac{\sin[(n+2)\cos^{-1}(\lambda/4\alpha)^{1/2}]}{\sin[\cos^{-1}(\lambda/4\alpha)^{1/2}]}.$$

Since all determinantal divisors d_i corresponding to H_n are equal to one except $d_n = \det(H_n - \lambda I_n)$, the minimum polynomial of H_n is $P_n(\lambda)$. Suppose

$$H_n = p^{-1}\begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix} P,$$

where B is nonsingular and N is nilpotent. Since $\operatorname{Ind}(H_n) = [n/2]$, the minimum polynomial of B is $f(\lambda) = P_n(\lambda)/\lambda^{[n/2]}$ or

$$f(\lambda) = \begin{cases} -\alpha^{(n+1)/2} \dfrac{\sin[(n+2)\cos^{-1}(\lambda/4\alpha)^{1/2}]}{\sin[\cos^{-1}(\lambda/4\alpha)^{1/2}]}, & \text{if } n \text{ is odd}, \\[2ex] \alpha^{(n+1)/2} \dfrac{\lambda^{-1/2}\sin[(n+2)\cos^{-1}(\lambda/4\alpha)^{1/2}]}{\sin[\cos^{-1}(\lambda/4\alpha)^{1/2}]}, & \text{if } n \text{ is even}. \end{cases}$$

The degree of $f(\lambda)$ is $(n+1)/2$ or $n/2$ depending on whether n is odd or even. If $f(\lambda) = a_t\lambda^t + a_{t-1}\lambda^{t-1} + \cdots + a_1\lambda + a_0 = 0$, then set

$$g(\lambda) = \lambda^{-1} = -\frac{1}{a_0}(a_t\lambda^{t-1} + \cdots + a_2\lambda + a_1).$$

Therefore [9], if $h(\lambda) = \lambda^{[n/2]} g^{[n/2]+1}(\lambda)$, then $(H_n)^D = h(H_n)$. If $\alpha = 0$, $(H_n)^D = 0$.

Example. Consider

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where $\alpha = 1$ and $n = 3$.

The index of H is 1, so that

$$f(\lambda) = -\frac{\sin[5\cos^{-1}(\lambda/4)^{1/2}]}{\sin[\cos^{-1}(\lambda/4)^{1/2}]}.$$

If $\lambda = 4\cos^2(\theta)$, then

$$\begin{aligned} f(\lambda) &= -\frac{\sin(5\theta)}{\sin(\theta)} = -\frac{1}{\sin(\theta)}[8\cos^2(\theta)\sin(\theta) - 16\sin^3(\theta)\cos^2(\theta) - 3\sin(\theta) + 4\sin^3(\theta)] \\ &= -(8\cos^2(\theta) - 16\sin^2(\theta)\cos^2(\theta) - 3 + 4\sin^2(\theta)). \end{aligned}$$

Upon substitution of $\cos(\theta) = (\lambda/4)^{1/2}$, $\sin(\theta) = ((4-\lambda)/4)^{1/2}$, $f(\lambda) = -(\lambda^2 - 3\lambda + 1)$ is the minimum polynomial for B. Therefore, $h(\lambda) = \lambda g^2(\lambda) = \lambda(3-\lambda)^2 = 9\lambda - 6\lambda^2 + \lambda^3$ and

$$H^D = h(H) = 9H - 6H^2 + H^3 = \begin{bmatrix} 2 & 2 & -3 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}.$$

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