32 [8].—THE INSTITUTE OF MATHEMATICAL STATISTICS, Editors, and H. L. HARTER & D. B. OWEN, Coeditors, *Selected Tables in Mathematical Statistics,* Volume I, American Mathematical Society, Providence, R. I., second printing with revisions, 1973, 403 pp., 26 cm. Price $8.60.

This is the first of a series of specialized tables prepared and edited by the Committee on Mathematical Tables of the Institute of Mathematical Statistics and published by the American Mathematical Society under a joint agreement.

The present volume contains five sets of tables; namely, "Tables of the Cumulative Noncentral Chi-Square Distribution", by G. E. Haynam, Z. Govindarajula and F. C. Leone, "Tables of the Exact Sampling Distribution of the Two-Sample Kolmogorov-Smirnov Criterion D_{mn}, $(m \leqslant n)$", by P. J. Kim and R. I. Jennrich, "Critical Values and Probability Levels for the Wilcoxon Rank Sum Test and the Wilcoxon Signed Rank Test," by Frank Wilcoxon, S. K. Katti and Roberta A. Wilcox, "The Null Distribution of the First Three Product-Moment Statistics for Exponential, Half-Gamma, and Normal Scores," by P. A. W. Lewis and A. S. Goodman, and "Tables to Facilitate the Use of Orthogonal Polynomials for Two Types of Error Structures," by Kirkland B. Stewart.

Each set of tables is prefaced by an introduction, a description of the mathematical algorithms used in their preparation, a discussion of tabular accuracy and interpolation, examples of their application, and references to the relevant literature.

A possible criticism of this collection of tables is that it is too specific and selective; however, this reviewer believes that this selection reflects the fact that most statistical texts do not adequately address the problems to which these tables apply. For example, this reviewer has many times been confronted with problems in chemical and mechanical engineering where a cumulative-error model beyond that of the first degree would have been appropriate, but necessary guidance was not to be found in the available literature. The tables of Stewart would have been extremely useful in that connection, and it is to be hoped that these tables will inspire similar research with other types of error structures.

Similarly, the tables of Lewis and Goodman address certain reliability problems involving failure clustering patterns that do not conform to typical textbook problems.

Especially useful is the presentation by Haynam, Govindarajula and Leone of two types of tables displaying different aspects of the power of the chi-square distribution as illustrated by well chosen examples.

The importance of the tables of Kim and Jennrich and also of those of Wilcoxon, Katti and Wilcox cannot be overemphasized for those researchers who depend upon distribution-free statistics for the solution of many of their problems.

In conclusion, this reviewer endorses this approach adopted by the Institute of Mathematical Statistics of soliciting meritorious material for mathematical statistical tables. This procedure should lead to a broad representation of those difficult statistical problems that continue to challenge researchers, and it should provide relevant tables not hitherto accessible in the literature.

HARRY FEINGOLD

33 [9].—I. O. ANGELL, *Table of Complex Cubic Fields,* Royal Holloway College, University of London, Surrey, England, 1972, 53 computer output sheets deposited in the UMT file.

There are listed here the 3169 nonconjugate cubic fields $Q(x)$ having discriminants $-D$ between 0 and -200000. For each $Q(x)$ there is given: D; a generating equation

$$x^3 - Ax^2 + Bx - C = 0$$

of discriminant $-N^2D$ and index N; the fundamental unit $\epsilon_0 = (Ix^2 + Jx + K)/L$ where $0 < \epsilon_0 < 1$ and L is a divisor of N; the class number H and an ideal norm bound P used in its calculation. The H and ϵ_0 are computed by Voronoi's method. This table should be useful and informative for all students of algebraic number theory.

There were two discrepancies between Angell's very brief paper [1] and the original table deposited in the Royal Society UMT. The paper states that $C(2) \times C(4)$ is the class group for $D = 16871$ while the table correctly had $H = 4$ since the group is really $C(2) \times C(2)$. The original table listed 3168 fields since one line was inadvertently omitted. It is included in the present version and is

D	N	A	B	C	I	J	K	L	P	H
5359	3	14	61	39	−17	236	−171	3	1	1

There follows a detailed critique of the conventions adopted in this table and then some further commentary and additions going beyond the table. In the equation selected for generating x, the coefficients A, B, C are all positive and such that the single real root satisfies $0 < x < 1$. That can always be accomplished simply by a translation $x = y + a$ or a reflection $x = a - y$. While this standardization certainly has merit it also has various minor faults: the coefficients are sometimes unduly large, and in further calculations it is frequently preferable to have the inflection point or any minimum of the cubic polynomial closer to $x = 0$. To illustrate, one of the 13 fields for $D = 63199139$, (far beyond this table), is generated by $y^3 - 183y^2 + 119y - 22 = 0$. By $y = 183 - x$, this becomes $f(x) = x^3 - 366x^2 + 33608x - 21755 = 0$ in Angell's convention. It has a minimum with $f(x) = \ldots 508, 85, 22, 325, 1000, \ldots$ far out at $x = 183$ while $f(x)$ has five or six decimals near $x = 0$. A more serious objection concerns the index N. It never exceeds 5 here but is not always minimized, not even when this can easily be done. For example, for the $D = 5359$ above, the $N = L = 3$ there can be eliminated since $x = (y + 1)/(y + 4)$ gives $y^3 - 59y - 175 = 0$ with $N = L = 1$. A minimal N is certainly preferable, both for practical computation and for theoretical studies concerning monogenic rings of integers, and, if and when N can be easily reduced, it seems desirable to do so.

The first five $N > 1$ here are for $D = 356, 424, 431, 440$, and 503, all being listed as $N = 2$. But, while 431 and 503 cannot be reduced to $N = 1$ since the prime 2 splits in these fields, the other three D can easily be reduced to $N = 1$. The next $N = 2$ is $D = 516$, but here I am uncertain whether it can be reduced or not. The first $N = 3$ here is for $D = 972$ (for $Q(\sqrt[3]{12})$) and this can be easily made $N = 1$. While $D = 2028$, for $Q(\sqrt[3]{26})$, can be easily reduced from $N = 3$ to $N = 2$, I am uncertain if it can be further reduced to $N = 1$. The first $N = 4$ and 5 here can be reduced to $N = 2$, and so forth.

The convention $0 < \epsilon_0 < 1$ also has a mixed assessment. Its reciprocal $\epsilon = \epsilon_0^{-1} > 1$ generally has much larger coefficients but ϵ can be used to easily compute the regulator $R = |\log \epsilon_0|$. In contrast, ϵ_0 may be exceedingly small and one has catastrophic loss of significance due to cancellation in its numerical evaluation unless one first inverts it *algebraically*. The programmer could circumvent this difficulty by printing ϵ_0 and evaluating and printing R in addition. (And why not? R is just as significant as H is.)

Davenport and Heilbronn have proven [2] that the asymptotic density of nonconjugate complex cubic fields is

$$[4\zeta(3)]^{-1} = 0.20798.$$

In this table one has an average density of only 3169/20000 = 0.15845. The ratio of the number of fields up to $D = 1000n$ divided by $1000n$ for $n = 1(1)20$ is shown in Table 1.

TABLE 1

n	*ratio*	*n*	*ratio*	*n*	*ratio*	*n*	*ratio*	*n*	*ratio*
1	0.1270	5	0.1458	9	0.1513	13	0.1544	17	0.1569
2	0.1350	6	0.1480	10	0.1520	14	0.1557	18	0.1572
3	0.1397	7	0.1514	11	0.1539	15	0.1561	19	0.1577
4	0.1435	8	0.1501	12	0.1540	16	0.1562	20	0.1584

The observed convergence is slowly from below and surprisingly smooth—except for a fluctuation at $D \approx 7000$–8000.

The growth here is associated mostly with those D for which m (> 1) nonconjugate fields exist. There are 58 D here with $m = 3$ and 22 with $m = 4$. $m > 4$ does not occur here. However, for larger D, there will be cases of $D = 27S^2$ where S is a square-free product of many primes. The multiplicity m increases exponentially with the number of prime factors of S. For fundamental discriminants $-D$, $m = (3^r - 1)/2$ where r is the 3-rank of $Q(\sqrt{-D})$. The 22 cases of $m = 4$ here are all of this type with $r = 2$. The maximum r known at present [3] is $r = 4$ and so its $D = 87386945207$ will have $m = 40$.

Most of the $m = 4$ discriminants here were already well-known, such as $D = 3299$, 4027, etc. Here are three known algebraic series [4], [5] that have $r \geqslant 2$ and therefore $m \geqslant 4$. These are the fundamental discriminants $-D$ where D equals

$$3\Delta(a, b) = 3(a^6 + 4b^6), \qquad b \equiv 0 \pmod 3,$$
$$D_6(z) = 108z^4 - 148z^3 + 84z^2 - 24z + 3, \qquad z(\neq 1) \equiv 1 \pmod 3,$$
$$4D_3(y) = 108y^4 - 296y^3 + 336y^2 - 192y + 48, \qquad y \equiv -1 \pmod 6.$$

For these D, N, A, B, C can be given a priori. Since I have not published this elsewhere, I include these formulas in Table 2. Note that the C in the first fields for $D_6(z)$ and $4D_3(y)$ are integral even when $z \equiv -1 \pmod 3$ and $y \equiv +1 \pmod 6$. In these cases

TABLE 2

D	N	A	B	C
$3\Delta(a, b)$	3	0	$3ab$	$2b^3 - a^3$
$3\Delta(a, b)$	3	0	$-3ab$	$2b^3 + a^3$
$3\Delta(a, b)$	3	0	$3b^2$	a^3
$3\Delta(a, b)$	6	0	$3a^2$	$4b^3$
$D_6(z)$	1	1	$1 - z$	$z(1 - 2z)$
$D_6(z)$	1	0	$-z$	$(6z^2 - 4z + 1)/3$
$D_6(z)$	1	0	$z(2 - 3z)$	$(6z^3 - 6z^2 + 4z - 1)/3$
$D_6(z)$	8	0	$8z - 3$	$(48z^2 - 40z + 10)/3$
$4D_3(y)$	2	1	$3 - 2y$	$4y^2 - 6y + 3$
$4D_3(y)$	2	0	$-2y$	$4y^2 - 8(2y - 1)/3$
$4D_3(y)$	2	0	$y(4 - 3y)$	$2y^3 - 4y^2 + 8(2y - 1)/3$
$4D_3(y)$	2	0	$4y - 3$	$4y^2 - 10(2y - 1)/3$

one only knows that $r \geqslant 1$ and $m \geqslant 1$, and these are valid cubic fields. Note also that the A, B, C in Table 2 do not follow Angell's convention.

There are known cases of $r = 3, 4$ in these series, such as the $D_6(28) = 63199139$ above, but they are far beyond Angell's table. Recently, F. Diaz y Diaz [6] sent me smaller D with $r = 3$. Two of these are

$$Q(\sqrt{-3321607}) \quad \text{with} \quad C(3) \times C(3) \times C(63),$$
$$Q(\sqrt{-3640387}) \quad \text{with} \quad C(3) \times C(3) \times C(18).$$

For these D an algebraic evaluation of the A, B, C for the 13 cubic fields is not possible and one must use numerical methods. By a delightfully sophisticated combination of the infrastructure [7] of the real fields $Q(\sqrt{3D})$ and unimodular and Tschirnhausen transformations, I computed the 13 cubic equations for these two D.

Table 3 shows the 13 fields for $D = 3321607$ with the A, B, C following Angell's convention. The splitting primes 2, 13, 19, 29, 41 and 43 split only in those four of the 13 fields marked S. This shows that the 13 fields are distinct and that I managed to make $N = 1$ except where 2 splits. In evaluating these cubic polynomials for $x = \pm 1$, ± 2, etc., one is struck with the large number of functional values equal to perfect cubes. These occur because these *cubic* fields have a 3-rank = 2 according to the Gras-Callahan theorem, cf. [8, p. 185].

TABLE 3, $D = 3321607$

N	A	B	C	2	13	19	29	41	43
8	45	664	404	S		S			
8	41	616	512	S	S				
8	59	960	656	S				S	S
6	37	498	288	S			S		
1	68	1179	755		S		S	S	
1	80	1601	27		S				
1	129	4174	883		S	S			S
1	2	95	27						S
1	17	144	125					S	
1	45	526	357			S	S		
1	9	112	103						
1	78	1555	1303			S		S	
1	126	4027	3637				S		S

For $D = 3640387$, no prime < 13 splits and $Q(\sqrt{-D})$ has $L(1, \chi) = 0.26674$. This is sufficiently close to the lower bound allowed by the Riemann Hypothesis [9] that it is unlikely that a much smaller quadratic class number than its $h = 27 \cdot 6$ can occur with $r \geqslant 3$. For this D, I found ten fields with $N = 1$, two with $N = 8$ and one with $N = 7$. I leave it as an exercise for the reader to reproduce these equations and to verify that 13 of the splitting primes and the 13 fields form the incomplete balanced block design [10] in Table 4. Any two fields intersect in only one of these splitting primes and any two primes both split in only one of these fields. Also, show that 149 splits in all 13 fields and 421 ramifies.

TABLE 4, $D = 3640387$

	13	31	43	53	73	109	173	193	227	239	281	337	617
I		S					S	S				S	
II			S	S	S							S	
III										S	S	S	S
IV	S					S			S			S	
V	S		S				S				S		
VI					S	S	S			S			
VII				S			S		S				S
VIII		S		S		S					S		
IX	S	S			S								S
X		S	S						S	S			
XI			S			S		S					S
XII					S			S	S		S		
XIII	S			S				S		S			

Since the infrastructure-Tschirnhausen method is quite efficient, and does not require much trial-and-error, one does not need a high-speed computer for D of this size, and I worked out these equations on a *nonprogrammable* HP-45 hand computer. One principal feature of the method is that as each cubic equation comes forth there is no need to show that it gives a field different than the others. That is automatic. I may publish this method elsewhere.

D. S.

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