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Numerical Solution for Eigenvalues and Eigenfunctions of a Hermitian Kernel and an Error Estimate

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Abstract. New error estimates for eigenvalues of symmetric integral equations are obtained. These estimates are applicable to a more general class of integration methods and, in many cases, are better than those of Wielandt. For every eigenvalue, a numerical solution for the corresponding eigenfunction is also obtained. Whenever the exact eigenvalue happens to be simple, an error estimate for the corresponding eigenfunction is also derived.

1. Introduction. Let $K(x, t)$ be a Hermitian kernel defined in $I \times I$, where $I \equiv [a, b]$, i.e., $K(t, x) = \overline{K(x, t)}$, such that

$$F(x) \equiv \int_a^b |K(x, t)|^2 \, dt \quad \text{is bounded in } I;$$

then all the *characteristic values* μ_i of $K(x, t)$ are real and there exists an orthonormal set $\{y_i(x)\}$ of *characteristic functions* [5], i.e.,

$$\int_a^b K(x, t) y_i(t) \, dt = \mu_i y_i(x), \qquad (y_i, y_j) = \delta_{ij}, \tag{1}$$

where $(u, v) \equiv \int_a^b u(x)\overline{v(x)}\, dx$ is the scalar product of two complex functions $u(x)$, $v(x) \in L_2(I) \equiv \{z(x) \mid (z, z) < \infty\}$.

Further, let S be a rule of numerical integration with weights $w_{in} > 0$ and nodes $x_{in} \in I$, $i = 1, \ldots, n$, by which the approximation $\int_a^b f(x)\, dx \approx \sum_{i=1}^n w_{in} f(x_{in})$ is made.

To obtain a numerical solution for the characteristic values of $K(x, t)$, Wielandt [9] replaced the original problem by the sequence of eigenproblems

$$K^{(n)} y_i^{(n)} = \mu_{in} y_i^{(n)}, \qquad K_{ij}^{(n)} \equiv w_{jn} K(x_{in}, x_{jn}), \qquad i, j = 1, \ldots, n, \tag{2}$$

with real μ_{in} and n linearly independent eigenvectors $y_i^{(n)}$, for a class of integration rules possessing the properties

$$\lim_{n \to \infty} \sum_{i=1}^n w_{in} f(x_{in}) = \int_a^b f(x) dx \quad \text{for every } f(x) \in C(I),$$

$$\sum_{i=1}^n w_{in} = b - a; \tag{3}$$

the eigenvalues μ_{kn}, $k = 1, \ldots, n$, are then taken by Wielandt as approximations, which also converge as $n \to \infty$, to the corresponding characteristic values of $K(x, t)$. To specify this correspondence, the following assumptions are made:

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Let $V \equiv \{\alpha_1, \ldots, \alpha_m\}$ be a subset of the set R of all eigenvalues of a square matrix A or of all characteristic values of a kernel $F(x, t)$ defined in $I \times I$, and let $W \equiv \{z^2 | z \in V\}$; then

(a) if $\alpha_1, \ldots, \alpha_m$ are the m largest (smallest) real elements of R such that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m$ $(\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m)$, then every $\alpha_i \neq \alpha_m$ with multiplicity $r_i > 1$ occurs r_i times in V,

(b) if $\alpha_1, \ldots, \alpha_m$ are the m real elements of R the largest modulus such that $|\alpha_1| \geqslant |\alpha_2| \geqslant \ldots \geqslant |\alpha_m|$ and there are r_i real elements of R of modulus $|\alpha_i|$, then every $\alpha_i^2 \neq \alpha_m^2$ occurs r_i times in W.

The problem which arises now is what is the best error estimate for the eigenvalues μ_{kn} of (2). In this context, and with the above assumptions, Wielandt obtained for those integration rules, which we shall call *convergent with respect to* $K(x, t)$–i.e. the sequence

$$\eta_n(x, t) \equiv \sum_{i=1}^{n} w_{in} K(x, x_{in}) K(x_{in}, t) - \int_a^b K(x, z) K(z, t)\, dz \tag{4}$$

of the error functions for $f(z) \equiv K(x, z)K(z, t)$ converges to 0 uniformly in $I \times I$, the following result:

Let $\mu_{1n}^+ \geqslant \mu_{2n}^+ \geqslant \ldots \geqslant \mu_{rn}^+ > 0 > \mu_{sn}^- \geqslant \ldots \geqslant \mu_{2n}^- \geqslant \mu_{1n}^-$ be the r largest positive and the s smallest negative eigenvalues of (2), and let

$$\mu_1^+ \geqslant \mu_2^+ \geqslant \ldots \geqslant \mu_r^+ > 0 > \mu_s^- \geqslant \ldots \geqslant \mu_2^- \geqslant \mu_1^-$$

be the corresponding characteristic values of $K(x, t)$; then

$$\mu_i^+ = \lim_{n\to\infty} \mu_{in}^+, \qquad \mu_j^- = \lim_{n\to\infty} \mu_{jn}^-, \qquad i = 1, \ldots, r,\ j = 1, \ldots, s,$$

and this convergence is uniform in i and j, i.e.,

$$\sigma_{kn} \equiv |\mu_{kn} - \mu_k| \leqslant q_n, \qquad \lim_{n\to\infty} q_n = 0,$$

where either

$$\mu_{kn} = \mu_{kn}^+, \qquad \mu_k = \mu_k^+, \quad \text{or} \quad \mu_{kn} = \mu_{kn}^-, \qquad \mu_k = \mu_k^-.$$

Baker [2] obtained convergence properties of a similar type for simple characteristic values of $K(x, t)$. The best estimate obtained by Wielandt is $q_n = O(\sqrt{\epsilon_n})$, where $\epsilon_n \equiv \max_{I\times I} |\eta_n(x, t)|$ and $\eta_n(x, t)$ is defined by (4), whereas that of Baker is $q_n = O(\max w_{in})$. Other authors ([1], [3]) obtained better bounds, but only for the distance of every eigenvalue μ_{kn} to the nearest characteristic value of $K(x, t)$. In this paper improved estimates of the form (see Theorem 1 at the beginning of Section 4)

$$\sigma_{kn} = [\max(|\mu_{kn}|, |\mu_k|)]^{-1} \rho_n, \qquad \rho_n = O(\epsilon_n),$$

are obtained, which generalizes Wielandt's convergence theorems for all integration rules which are convergent with respect to $K(x, t)$ and satisfy (3). Moreover, the new result enables application of integration rules, which are convergent with respect to $K(x, t)$, to kernels which exhibit singular behavior in $I \times I$ and for which, therefore, no solution can be found within the scope of Wielandt's and Baker's papers (see Example 2 in Section 2). As a consequence, an error estimate for the numerical solution of (1), convergent to 0 uniformly in I for every integration rule which is convergent with respect to $K(x, t)$, is derived. The new error estimates for eigenvalues are to be interpreted as follows: our estimates are better than those of Wielandt for the first m_n eigenvalues μ_{kn} such that $\max(|\mu_{kn}|, |\mu_k|) > C\sqrt{\epsilon_n}$ for some $C > 0$, where the sequence m_n tends to infinity; for other eigenvalues both our and Wielandt's estimates are of the same order of magnitude, namely $O(\sqrt{\epsilon_n})$, and ours are not necessarily better.

2. Numerical Results. To illustrate the superiority of the new error estimates given by Theorem 1 in Section 4, two numerical examples are presented. To the second of our examples Wielandt's method does not apply.

In the tables of results given below, μ_{ln}^+ and μ_{ln}^- are the eigenvalues defined in Theorem 2 near the end of Section 4, whereas $y_{ln}^+(x)$ and $y_{ln}^-(x)$ are the numerical solutions for characteristic functions corresponding to μ_{ln}^+ and μ_{ln}^-, respectively, obtained by the procedure described at the end of Section 4. The improved error estimates are those described in Section 5. The error estimates for $y_{ln}^+(x)$ and $y_{ln}^-(x)$ are those obtained by application of the remark concluding the discussion of Theorem 3 in Section 4.

Example 1. The integral equation is

$$\int_0^1 \max(x, t)y(t)\, dt = \mu y(x).$$

Characteristic values and characteristic functions are, respectively:

R^2 and $\dfrac{\sqrt{2}\cosh Rx}{\cosh R}$, where R is the positive root of the equation $z \tanh z = 1$;

$-r_N^2$ and $\dfrac{\sqrt{2}\cos r_N x}{\cos r_N}$, $N = 1, 2, \ldots$, where $0 < r_1 < r_2 < \ldots$ are the positive roots of the equation $z \tan z + 1 = 0$.

The integration rule S mentioned at the introduction is the trapezoidal rule.

To obtain α_n, β_n and γ_n, as defined in (24), put

$$A_n(z) \equiv (n-1)z - [(n-1)z], \qquad B_n(z) \equiv 1 - A_n(z), \qquad C_n(z) \equiv A_n(z)B_n(z),$$

$$D_n(z) \equiv A_n(z) - B_n(z), \qquad h \equiv (n-1)^{-1}, \qquad F_n(z) \equiv 1 - z - C_n(z)[3z - hD_n(z)];$$

then

$$\eta_n(x, t) = \frac{h^2}{6}\begin{cases} 3tC_n(x) + F_n(t), & x \leqslant t, \\ 3xC_n(t) + F_n(x), & x \geqslant t, \end{cases}$$

which after a simple, but lengthy, calculation yields

$$\alpha_n^2 = \frac{h^4}{18}\sum_{k=1}^{n-1}\int_{(k-1)h}^{kh} t\{tG_n(t) + hC_n(t)D_n(t)[G_n(t) + 0.3tC_n(t)] + F_n^2(t)\}\, dt,$$

where $G_n(t) \equiv 0.3t + F_n(t)$, and

$$\int_0^1 \eta_n^2(x, x_{in})\, dx = \int_0^{x_{in}} \eta_n^2(x, x_{in})\, dx + \int_{x_{in}}^1 \eta_n^2(x, x_{in})\, dx.$$

Each of the above summands is evaluated by the closed Newton-Cotes formula with 7 points. The remark at the end of Section 4 is applied with $p = L = 1$.

For comparison with Wielandt's results the error estimates for the negative eigenvalues μ_{ln}^-, together with error estimates for the numerical solutions for characteristic functions, are presented in the following table:

TABLE 1

Case	l	*Error estimate for* μ_{ln}^- *by Theorem* 1	*Improved error estimate for* μ_{ln}^-	*Error estimate for* $y_{ln}^-(x)$	*Actual error for* μ_{ln}^-	*Actual maximal error for* $y_{ln}^-(k/N)$, $k = 0, 1, \ldots, N$	*Wielandt's estimate*
$n = 101$	1	$7.34 \cdot 10^{-5}$	$7.34 \cdot 10^{-5}$	0.0067	$1.26 \cdot 10^{-5}$	0.00011	
	2	$6.45 \cdot 10^{-4}$	$3.51 \cdot 10^{-4}$	0.872	$9.22 \cdot 10^{-6}$	0.000466	0.00539
$N = 200$	3	0.00153	$8.15 \cdot 10^{-4}$		$8.72 \cdot 10^{-6}$	0.00104	
	1	$1.836 \cdot 10^{-5}$	$1.836 \cdot 10^{-5}$	0.00167	$3.14 \cdot 10^{-6}$	$2.77 \cdot 10^{-5}$	
$n = 201$ $N = 1000$	2	$1.61 \cdot 10^{-4}$	$8.78 \cdot 10^{-5}$	0.2073	$2.306 \cdot 10^{-6}$	$1.17 \cdot 10^{-4}$	0.00269
	3	$3.75 \cdot 10^{-4}$	$2.04 \cdot 10^{-4}$		$2.18 \cdot 10^{-6}$		
	4	$6.85 \cdot 10^{-4}$	$3.66 \cdot 10^{-4}$		$1.628 \cdot 10^{-5}$		

It is to be noted that as the initial error estimates for the eigenvalues μ_{ln}^- tend to grow with l, they are better than those of Wielandt only for some first eigenvalues. To obtain a comparable error estimate unobtainable by Theorem 1 for other eigenvalues, the bound q_n defined by (26) with the optimal $C = \sqrt{0.5(1 + \sqrt{5})}$ can be taken.

Example 2. The integral equation is

$$\int_0^1 (1 + i\sqrt{x} - i\sqrt{t})^{-1} y(t)\, dt = \mu y(x).$$

The exact solution is unknown.

The integration rule, which is derived by the transformation $u = z^2$ for the integral $\int_0^1 K(x, z)K(z, t)\, dz$ and application of the Gauss quadrature with weights ω_{in} and nodes ξ_{in}, $i = 1, \ldots, n$, is defined by

$$w_{in} \equiv 2\omega_{in}\xi_{in}, \qquad x_{in} \equiv \xi_{in}^2, \qquad i = 1, \ldots, n;$$

therefore, using our definition (4) [7, p. 48],

$$\eta_n(x, t) = 2c_n \left[\frac{\partial^{2n}}{\partial u^{2n}} uK(x, u^2)K(u^2, t)\right]_{u=\xi}$$
$$= \frac{2(2n)!c_n}{\sqrt{x} - \sqrt{t} - 2i} \left[\frac{\sqrt{x} - i}{(\xi - \sqrt{x} + i)^{2n+1}} - \frac{\sqrt{t} + i}{(\xi - \sqrt{t} - i)^{2n+1}}\right],$$

where $c_n = [\binom{2n}{n}^2(2n + 1)!]^{-1}$ and $0 < \xi = \xi(x, t) < 1$, and consequently

$$|\eta_n(x, t)| \leqslant \binom{2n}{n}^{-2}(2n + 1)^{-1}(\sqrt{1 + x} + \sqrt{1 + t}).$$

The error estimates, with those for $y_{ln}^+(x)$ obtained by application of the remark at the end of Section 4 with $p = ½$ and $L = 1$, are given in the table below:

TABLE 2

n	l	*Error estimate for* μ_{ln}^+ *by Theorem* 1	*Improved error estimate for* μ_{ln}^+	*Error estimate for* $y_{ln}^+(x)$
6	1	$2.31 \cdot 10^{-7}$	$2.31 \cdot 10^{-7}$	$8.8 \cdot 10^{-7}$
	2	$1.015 \cdot 10^{-5}$	$5.08 \cdot 10^{-6}$	0.00383
	3	$2.12 \cdot 10^{-4}$	$1.033 \cdot 10^{-4}$	
8	1	$9.1 \cdot 10^{-10}$	$9.1 \cdot 10^{-10}$	$3.41 \cdot 10^{-9}$
	2	$4 \cdot 10^{-8}$	$2 \cdot 10^{-8}$	$1.47 \cdot 10^{-5}$
	3	$8.15 \cdot 10^{-7}$	$4.08 \cdot 10^{-7}$	0.11

The approximations for $n = 9$ rounded to 10 digits are:

$$\mu_{1n}^+ \approx 0.9543482459, \quad \mu_{2n}^+ \approx 0.0434068611, \quad \mu_{3n}^+ \approx 0.0021321407.$$

3. Numerical Solution for a Characteristic Function. Since the new results, presented in Section 2, refer also to an error estimate for the corresponding characteristic function, an appropriate definition of the approximate solution for a characteristic function, which converges to the corresponding characteristic function, is to be given. To obtain such a definition, observe first that by the similarity relation between the matrix $K^{(n)}$ of (2) and the Hermitian matrix H with $H_{ij} \equiv K(x_{in}, x_{jn})\sqrt{w_{in}w_{jn}}$, $i, j = 1, \ldots, n$, the eigenvector $y_k^{(n)}$ is related to the corresponding eigenvector z_k of H by $z_{ki} = y_{ki}^{(n)}\sqrt{w_{in}}$, $i = 1, \ldots, n$. Further, define a new scalar product $(u, v)_n$ of two vectors u, v in C_n—the n-dimensional complex Euclidean space—and a new norm $|u|_n$ in C_n, by

$$(u, v)_n \equiv \sum_{i=1}^{n} w_{in} u_i \bar{v}_i, \qquad |u|_n \equiv \sqrt{(u, u)_n}, \tag{5}$$

and denote by $\|f\| \equiv \sqrt{(f, f)}$ the norm of a complex function $f(x)$; therefore, if the eigenvectors z_k, $k = 1, \ldots, n$, of H are chosen so as to form an orthonormal set, then

$$(y_p^{(n)}, y_q^{(n)})_n = \delta_{pq}, \qquad p, q = 1, \ldots, n. \tag{6}$$

For every eigenvector $y_k^{(n)}$ of (2) with $\mu_{kn} \neq 0$, define now the *numerical solution* $y_{kn}(x)$ *for a characteristic function generated by* $y_k^{(n)}$, which also satisfies $y_{kn}(x_{in}) = y_{ki}^{(n)}$, $i = 1, \ldots, n$, as

$$y_{kn}(x) \equiv \mu_{kn}^{-1} \sum_{j=1}^{n} w_{jn} y_{kj}^{(n)} K(x, x_{jn}). \tag{7}$$

It is natural to expect the difference between the two sides of (1), with μ_k and $y_k(x)$ replaced by μ_{kn} and $y_{kn}(x)$, respectively, to be expressible in terms of the error function (4). In fact,

$$\mu_{kn} y_{kn}(x) - \int_a^b K(x, t) y_{kn}(t)\,dt = \mu_{kn}^{-1} \sum_{j=1}^{n} w_{jn} y_{kj}^{(n)} \eta_n(x, x_{jn}), \tag{8}$$

where $\eta_n(x, t)$ is defined by (4).

Let, further, $\{y_m^*(x)\}$, $m = 1, \ldots, r$, form an orthonormal base of all characteristic functions of $K(x, t)$ corresponding to μ_k; then for every n with $\mu_{kn} \neq 0$, there exist coefficients $c_{km}^{(n)}$, $m = 1, \ldots, r$, such that the error function

$$e_{kn}(x) \equiv y_{kn}(x) - \sum_{m=1}^{r} c_{km}^{(n)} y_m^*(x)$$

is of minimal norm. In fact,

$$c_{km}^{(n)} = (y_{kn}, y_m^*), \qquad m = 1, \ldots, r;$$

and, consequently,

(9) $(e_{kn}, y) = 0$ for every characteristic function $y(x)$ of $K(x, t)$ corresponding to μ_k.

The functions $e_{kn}(x)$ and $\tilde{y}_{kn}(x) \equiv y_{kn}(x) - e_{kn}(x)$ are called *the error function and the characteristic function,* respectively, *associated with* $y_{kn}(x)$. Now, if the approximate numerical solution $y_{kn}^*(x)$ for a characteristic function is taken to be of norm 1, i.e., $y_{kn}^*(x) = \|y_{kn}\|^{-1} y_{kn}(x)$, it can be shown that the characteristic function $Y_{kn}(x)$ of norm 1 corresponding to μ_k such that the error function $e_{kn}^*(x) \equiv y_{kn}^*(x) - Y_{kn}(x)$ is of minimal norm, assumes the form

$$Y_{kn}(x) = \begin{cases} R_{kn}^{-1} \tilde{y}_{kn}(x), & R_{kn} \neq 0, \\ y_l(x) \quad \text{with } \mu_l = \mu_k, & R_{kn} = 0, \end{cases}$$

where $R_{kn} = \|\tilde{y}_{kn}\|$. Also, since by (9) $\|y_{kn}\|^2 = R_{kn}^2 + \|e_{kn}\|^2$, we have

$$e_{kn}^*(x) = \|y_{kn}\|^{-1}[e_{kn}(x) - (\|y_{kn}\| - R_{kn}) Y_{kn}(x)]$$

(10)
$$= \|y_{kn}\|^{-1} \left[e_{kn}(x) - \frac{\|e_{kn}\|^2}{\|y_{kn}\| + R_{kn}} Y_{kn}(x) \right].$$

4. Error Estimate and Convergence. For the sake of conciseness of presentation, the following definitions are introduced:

(11) μ_k and μ_{kn}, $k = 1, \ldots, r$, are the r largest (smallest) characteristic values of $K(x, t)$ and the r largest (smallest) eigenvalues of (2), respectively, such that $\mu_i \geqslant \mu_{i+1}$ $(\mu_i \leqslant \mu_{i+1})$ and $\mu_{in} \geqslant \mu_{i+1,n}$ $(\mu_{in} \leqslant \mu_{i+1,n})$, $i = 1, \ldots, r-1$.

In the following, $F(x, t)$ is a kernel defined in $I \times I$.

(12) $U(F)$ and $U_n(F)$ are the set of all characteristic values of $F(x, t)$ and the set of all eigenvalues of the matrix $F^{(n)}$ with $F_{ij}^{(n)} \equiv w_{jn} F(x_{in}, x_{jn})$, $i, j = 1, \ldots, n$, respectively.

(13) $\lambda_k(F)$ and $\lambda_{kn}(F)$ are the kth real elements of $U(F)$ and $U_n(F)$, respectively, in the ordering determined by that of the μ_k and the μ_{kn} in (11).

(14) $M_k(F)$ and $M_{kn}(F)$, $k = 1, \ldots, r$, are the moduli of the r elements of $U(F)$ and $U_n(F)$, respectively, of largest modulus, such that $M_i(F) \geqslant M_{i+1}(F)$ and $M_{in}(F) \geqslant M_{i+1,n}(F)$, $i = 1, \ldots, r-1$.

(15)
$$Q(F, u) \equiv \int_a^b \int_a^b F(x, t) u(t) \overline{u(x)}\, dx\, dt,$$

(16)
$$Q_n(F, u) \equiv \sum_{i,j=1}^{n} w_{in} w_{jn} F(x_{in}, x_{jn}) u_j \bar{u}_i,$$

(17) $$V_k(F) \equiv \left\{ Y \middle| \int_a^b F(x, t) Y(t) dt = \lambda_k(F) Y(x),\ \|Y\| = 1 \right\},$$

(18) $$V_{kn}(F) \equiv \{ z \mid F^{(n)} z = \lambda_{kn}(F) z,\ |z|_n = 1 \}, \quad \text{where}$$

$$F_{ij}^{(n)} \equiv w_{jn} F(x_{in}, x_{jn}),\ i, j = 1, \ldots, n, \quad \text{and } |z|_n \text{ is defined by (5).}$$

(19) $$\delta_n(F, x, t) \equiv \sum_{i=1}^n w_{in} F(x, x_{in}) F(x_{in}, t) - \int_a^b F(x, z) F(z, t)\, dz,$$

(20) $$D_n(F, u) \equiv \int_a^b \int_a^b \delta_n(F, x, t) \overline{u(x)} u(t)\, dx\, dt,$$

(21) $$D_n^*(F, u) \equiv \sum_{i,j=1}^n w_{in} w_{jn} \delta(F, x_{in}, x_{jn}) u_j \bar{u}_i,$$

(22) $$A_{kn}(F) \equiv \left[\max\left\{ \sum_{i=1}^n w_{in} \left| \int_a^b \delta_n(F, x, x_{in}) \overline{u(x)}\, dx \right|^2 \middle| u \in V_k(F) \right\} \right]^{1/2},$$

(23) $$B_{kn}(F) \equiv \left[\max\left\{ \int_a^b \left| \sum_{i=1}^n w_{in} \bar{u}_i \delta_n(F, x_{in}, x) \right|^2 dx \middle| u \in V_{kn}(F) \right\} \right]^{1/2};$$

$$\alpha_n \equiv \left[\int_a^b \int_a^b |\eta_n(x, t)|^2\, dx\, dt \right]^{1/2}, \quad \beta_n \equiv \left[\sum_{i,j=1}^n w_{in} w_{jn} |\eta_n(x_{in}, x_{jn})|^2 \right]^{1/2},$$

(24) $$\gamma_n \equiv \left[\sum_{i=1}^n w_{in} \int_a^b |\eta_n(x, x_{in})|^2\, dx \right]^{1/2}, \qquad \rho_n \equiv \max(\alpha_n, \beta_n),$$

where $\eta_n(x, t)$ is defined by (4).

(25) $y_{kn}(x)$ and $e_{kn}(x)$ are, respectively, the function (7) and the error function associated with it as defined in Section 3.

The new error estimates for the eigenvalues obtained in this paper are now summarized in the following two theorems:

THEOREM 1. *If, with Definitions* (11) *and* (24), $\nu_{kn} \equiv \max(|\mu_{kn}|, |\mu_k|) \geqslant C\sqrt{\rho_n}$ *for some* $C > 1$, *then*

(a) $$|\mu_{kn} - \mu_k| \leqslant \nu_{kn}^{-1}(\gamma_n + \rho_n)[1 - \nu_{kn}^{-2}\rho_n]^{-1/2} \leqslant \nu_{kn}^{-1}(\gamma_n + \rho_n)[1 - C^{-2}]^{-1/2},$$

(b) $$|\mu_{1n} - \mu_1| \leqslant \gamma_n[\nu_{1n}^2 - \rho_n]^{-1/2}.$$

THEOREM 2. *Let* $\mu_{1n}^+ \geqslant \mu_{2n}^+ \geqslant \ldots \geqslant \mu_{rn}^+ > 0$, $\mu_{1n}^- \leqslant \mu_{2n}^- \leqslant \ldots \leqslant \mu_{sn}^- < 0$, *be the r largest positive and the s smallest negative eigenvalues of* (2), *and let* $\mu_1^+ \geqslant \mu_2^+ \geqslant \ldots \geqslant \mu_r^+ > 0 > \mu_s^- \geqslant \ldots \geqslant \mu_2^- \geqslant \mu_1^-$ *be the corresponding characteristic values of* $K(x, t)$.

If the integration rule S is convergent with respect to $K(x, t)$ and satisfies (3), *then*

$$\lim_{n\to\infty} \mu_{in}^{+} = \mu_i^{+}, \qquad \lim_{n\to\infty} \mu_{jn}^{-} = \mu_j^{-}, \qquad i = 1, \dots, r,\ j = 1, \dots, s;$$

and the convergence is uniform in i and j, so that for every $C > 1$,

$$|\mu_{in}^{+} - \mu_i^{+}| \leqslant q_n, \qquad |\mu_{jn}^{-} - \mu_j^{-}| \leqslant q_n, \qquad i = 1, \dots, r,\ j = 1, \dots, s,$$

(26) $\quad q_n \equiv \max\{C, [C^2 - 1]^{-1/2}\}\sqrt{\gamma_n + \rho_n},$ *where γ_n and ρ_n are defined by* (24).

Theorem 2 is a generalization of Wielandt's results.

The error estimate for the approximate numerical solution of (1) is given by:

THEOREM 3. *The error function defined by* (25) *satisfies* (*see Definitions* (4), (11) *and* (12))

$$|e_{kn}(x)| \leqslant |\mu_k^{-1}\mu_{kn}^{-1}| \left\{ \sum_{j=1}^{n} w_{jn}\, |y_{kj}^{(n)}| \left[\max_I |\eta_n(x, x_{jn})| + q_{kn} I_{jn} \sqrt{F(x)} \right] + |\mu_{kn} - \mu_k| G_n(x) \right\},$$

where

$$F(x) \equiv \int_a^b |K(x, t)|^2 dt, \qquad G_n(x) \equiv [F(x) + \eta_n(x, x)]^{1/2},$$

(27) $$q_{kn} \equiv \sup\{|\mu_{kn} - \lambda|^{-1} \mid \lambda \in U(K), \lambda \neq \mu_k\},$$

$$I_{jn} \equiv \left[\int_a^b |\eta_n(x, x_{jn})|^2 dx \right]^{1/2}.$$

This bound for $e_{kn}(x)$, and consequently that for the function $e_{kn}^{*}(x)$ defined by (10), are improvements, by a factor of $O(n^{-1/2})$, of a similar error estimate obtained in [4].

Error estimates for the eigenvalues in special cases are given in Section 5.

An immediate consequence of Theorem 2, analogous to the one which follows from the convergence theorem in [1], is:

If the integration rule S is convergent with respect to $K(x, t)$, then $e_{kn}(x)$ and $e_{kn}^{*}(x)$ converge to 0 uniformly in I.

Remark. If $K(x, t)$ satisfies a Lipschitz condition of the form $|K(u, t) - K(v, t)| \leqslant L|u - v|^p$, $0 < p \leqslant 1$, in $I \times I$, then (see Definition (4))

$$|e_{kn}(x)| \leqslant |\mu_k^{-1}| \left\{ |\mu_{kn}^{-1}| \sum_{j=1}^{n} w_{jn}\, |y_{kj}^{(n)}| \left[\max_I |\eta_n(x, x_{jn})| + q_{kn} I_{jn} \sqrt{F(x)} \right] + |\mu_{kn} - \mu_k| \left[\max_m |y_{km}^{(n)}| + \frac{2^{-p} L}{|\mu_{kn}|} \max_{0 \leqslant m \leqslant n} (x_{m+1,n} - x_{mn})^p \sum_{j=1}^{n} w_{jn}\, |y_{kj}^{(n)}| \right] \right\}$$

where $x_{0n} = a$ and $x_{n+1,n} = b$.

Since the estimate for $e_{kn}(x)$ involves an estimate for q_{kn}, it can be found only if the multiplicity of μ_k is known. Such an estimate is obtainable, for instance, when μ_k is a simple characteristic value and in this case we deduce by Theorem 2, Eq. (38) of [6] and Lemma 1 in Section 6:

COROLLARY 1. *If μ_k is a simple characteristic value of $K(x, t)$ and the integration rule is convergent with respect to $K(x, t)$ and satisfies* (3), *then for some choice of eigenvectors $y_k^{(n)}$ such that $|y_k^{(n)}|_n = 1$ (see Definitions* (5) *and* (7)),

$$y_{kn}(x) \longrightarrow y_k(x) \quad \textit{uniformly in } I, \textit{ and so does also } \|y_{kn}\|^{-1} y_{kn}(x).$$

By (7) it follows that

$$\|y_{kn}\|^2 = \mu_{kn}^{-2} \sum_{i,j=1}^{n} w_{in} w_{jn} \overline{y_{ki}^{(n)} y_{kj}^{(n)}} G(x_{in}, x_{jn})$$

where

$$G(x, t) \equiv \int_a^b \overline{K(z, x)} K(z, t)\, dz = \int_a^b K(x, z) K(z, t)\, dz.$$

In the case where $G(x, t)$ cannot be determined exactly, an approximation c_{kn} of $\|y_{kn}\|$ is found by applying some quadrature formula for determining $G(x, t)$ at the points (x_{in}, x_{jn}); the approximate solution for a characteristic function is then taken to be $c_{kn}^{-1} y_{kn}(x)$, and the error estimate is

$$\widetilde{e}_{kn}(x) = |c_{kn}^{-1} y_{kn}(x) - R_{kn}^{-1} \widetilde{y}_{kn}(x)| \leqslant |(c_{kn}^{-1} - \|y_{kn}\|^{-1}) y_{kn}(x)| + |e_{kn}^*(x)|$$

$$= (c_{kn} \|y_{kn}\|)^{-1} |(c_{kn} - \|y_{kn}\|) y_{kn}(x)| + |e_{kn}^*(x)|,$$

where $e_{kn}^*(x)$ is given by (10).

5. Improved Error Estimates for Simple Characteristic Values and for Positive-Definite Kernels. An error estimate for a simple characteristic value can be improved if the approximate eigenvalue μ_{kn} satisfies the inequality (see Definition (11))

$$|\mu_{kn} - \mu_k| < \min_{i \neq k} |\mu_{kn} - \mu_i|.$$

If the integration rule S is convergent with respect to $K(x, t)$ and satisfies (3), and μ_k is a simple characteristic value, then by Theorem 2 there exists an integer N such that the above inequality holds for $n > N$, and by Lemma 5 (stated in Section 6)

$$|\mu_{kn} - \mu_k| = \inf\{|\mu_{kn} - \lambda| \mid \lambda \in U(K)\} \leqslant |\mu_{kn}^{-1}| \|y_{kn}\|^{-1} \gamma_n.$$

An error estimate for a positive-definite kernel is obtained from the following theorem:

THEOREM 4. *Let $\widetilde{\mu}_{kn} \in U_n(K)$ and $\widetilde{\mu}_k \in U(K)$, $k = 1, \ldots, n$, such that $|\widetilde{\mu}_{kn}| = M_{kn}(K)$ and $|\widetilde{\mu}_k| = M_k(K)$ (see Definitions* (12), (14) *and* (4)). *Then*

$$|\widetilde{\mu}_{kn}^2 - \widetilde{\mu}_k^2| \leqslant M_1(\eta_n) + M_{1n}(\eta_n), \qquad k = 1, 2, \ldots,$$

where $\widetilde{\mu}_{kn} = 0$ for $k > n$.

COROLLARY 2. *If $K(x, t)$ is positive-definite and (see Definitions* (11) *and* (12)), *$\mu_{kn} > -\min\{\lambda \mid \lambda \in U_n(K)\}$, then, $|\mu_{kn}^2 - \mu_k^2| \leqslant M_1(\eta_n) + M_{1n}(\eta_n)$.*

We also obtain

COROLLARY 3. $|\mu_{n+k}| \leqslant \sqrt{M_1(\eta_n)}$, $k = 1, 2, \ldots$.

6. Discussion of the Theorems. To obtain the final results presented in Theorems 1–4, the following lemmas are necessary using the definitions introduced at the beginning of Section 4:

LEMMA 1. *Let* $u_n = (u_{n1}, \ldots, u_{nn})$ *and* $A_n = (a_{ij}^{(n)})$ *be, respectively, sequences of vectors and* $n \times n$ *matrices with complex elements. Then* (*see Definition* (5))

(a) $|u_n|_n^2 = \Sigma_{k=1}^n |(u_n, z_k)_n|^2 = \Sigma_{k=1}^n |(z_k, u_n)_n|^2$ *for every sequence* z_k, $k = 1, \ldots, n$, *satisfying*

$$(z_p, z_q)_n = \delta_{pq}, \quad p, q = 1, \ldots, n. \tag{28}$$

(b) *If the integration rule S satisfies* (3), *then*

$$\lim_{n\to\infty} \max_i |u_{ni}| = 0 \textit{ implies } \lim_{n\to\infty} \sum_{i=1}^n w_{in} u_{ni} = \lim_{n\to\infty} \sum_{i=1}^n w_{in} |y_{ki}^{(n)} u_{ni}| = 0, \qquad k = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} \max_{i,j} |a_{ij}^{(n)}| = 0 \textit{ implies } \lim_{n\to\infty} \sum_{i,j=1}^n w_{in} w_{jn} a_{ij}^{(n)} = 0.$$

LEMMA 2. *If* $\lambda_k \equiv \lambda_k(F) = \lambda_1(F)$, *where* $F(x, t)$ *is a Hermitian kernel defined in* $I \times I$, *then* (*see Definitions* (13), (22), (17) *and* (20))

$$\lambda_k(\lambda_k - \lambda_{kn}(F)) \leqslant A_{kn}(F)\left[1 - \lambda_k^{-2} \max_{V_k(F)} |D_n(F, u)|\right]^{-1/2}.$$

LEMMA 3. *If* $\lambda_{kn} \equiv \lambda_{kn}(F) = \lambda_{1n}(F)$, *where* $F(x, t)$ *is a Hermitian kernel defined in* $I \times I$, *then* (*see Definitions* (13), (23), (18) *and* (21))

$$\lambda_{kn}(\lambda_{kn} - \lambda_k(F)) \leqslant B_{kn}(F)\left[1 - \lambda_{kn}^{-2} \max_{V_{kn}(F)} |D_n^*(F, u)|\right]^{-1/2}.$$

This lemma is a consequence of Weyl's theorem [8, p. 445]:

LEMMA 4. *Let* $D(x, t) \equiv F(x, t) - G(x, t)$, *where* $F(x, t)$ *and* $G(x, t)$ *are Hermitian kernels defined in* $I \times I$; *then* (*see Definitions* (13), (15) *and* (16))

(a) *if* $Q(D, u) \geqslant 0$ *for every* $u(x)$, *then* $\lambda_k(F) \geqslant \lambda_k(G)$, $k = 1, 2, \ldots$;

(b) *if* $Q_n(D, u) \geqslant 0$ *for every* $u \in C_n$, *then* $\lambda_{kn}(F) \geqslant \lambda_{kn}(G)$, $k = 1, \ldots, n$.

The next and last lemma is used to obtain the improved error estimates for simple characteristic values mentioned in Section 5.

LEMMA 5. *With Definitions* (11), (12), (7) *and* (24),

$$D_{kn} \equiv \inf\{|\mu_{kn} - \lambda| \,|\lambda \in U(K)\} \leqslant |\mu_{kn}^{-1}| \, \|y_{kn}\|^{-1} \gamma_n.$$

This is a slight improvement of the result obtained in [3].

The proofs of Lemmas 1, 4 and 5 are straightforward ([6, Lemmas 1, 5 and 2, respectively]), whereas those of Lemmas 2 and 3 require some special devices ([6, Lemmas 3 and 4, respectively]).

The first four lemmas are used to establish part (a) of Theorem 1 by the following steps:

1. Application of Lemma 2 and part (b) of Lemma 4 to obtain (see Definitions (11), (22), (17) and (20))

$$\mu_k(\mu_k - \mu_{kn}) \leqslant A_{kn}(L)\left[1 - \mu_k^{-2} \max_{V_k(L)} |D_n(L, u)|\right]^{-1/2}, \tag{29}$$

where

$$L(x, t) \equiv K(x, t) - \sum_{p=1}^{k-1} (\mu_p - \mu_k) y_p(x) \overline{y_p(t)}. \tag{30}$$

2. Application of Lemma 3 and part (a) of Lemma 4 to obtain (see Definitions (11), (23), (18) and (21))

$$\mu_{kn}(\mu_{kn} - \mu_k) \leqslant B_{kn}(L_n)\left[1 - \mu_{kn}^{-2} \max_{V_{kn}(L_n)} |D_n^*(L_n, u)|\right]^{-1/2}, \tag{31}$$

where

$$L_n(x, t) \equiv K(x, t) - \sum_{p=1}^{k-1} (\mu_{pn} - \mu_{kn}) y_{pn}(x) \overline{y_{pn}(t)}. \tag{32}$$

3. Bounding of $A_{kn}(L)$, $\max\{|D_n(L, u)| \mid u \in V_k(L)\}$, $B_{kn}(L_n)$ and $\max\{|D_n^*(L_n, u)| \mid u \in V_{kn}(L_n)\}$ in terms of γ_n and ρ_n defined by (24), which is a matter of pure manipulations.

Theorem 2 follows from Lemma 1 and Theorem 1.

Theorem 3 is a consequence of (9) and the Parseval equality (equation of closedness [5, p. 10]) for the function $e_{kn}(x)$.

For the full proof of the above theorems the reader is referred to [6].

Finally, we come to the proof of Theorem 4, which terminates our discussion.

Proof of Theorem 4. The degenerate kernel

$$G_n(x, t) \equiv \sum_{i=1}^{n} w_{in} K(x, x_{in}) K(x_{in}, t) = \sum_{i=1}^{n} w_{in} \overline{K(x_{in}, x)} K(x_{in}, t),$$

is Hermitian and $G_n(x, t) = G(x, t) + \eta_n(x, t)$, where

$$G(x, t) \equiv \int_a^b K(x, z) K(z, t)\, dz,$$

therefore the characteristic values ν_{kn} of $G_n(x, t)$, where $\nu_{1n} \geqslant \nu_{2n} \geqslant \ldots \geqslant \nu_{nn} \geqslant \nu_{n+1,n} = \ldots = 0$, are related to those of $G(x, t)$, which are $\widetilde{\mu}_k^2$, by the inequalities [8, p. 445]:

$$|\nu_{kn} - \widetilde{\mu}_k^2| \leqslant M_1(\eta_n), \qquad k = 1, 2, \ldots . \tag{33}$$

The ν_{kn}, $k = 1, \ldots, n$, are exactly the eigenvalues of the matrix $L_n \equiv (w_{in} G(x_{in}, x_{jn}))$, which is similar to the Hermitian matrix $A^{(n)}$ defined by

$$A_{ij}^{(n)} \equiv \sqrt{w_{in} w_{jn}}\, G(x_{in}, x_{jn}) = \sqrt{w_{in} w_{jn}}\,[G_n(x_{in}, x_{jn}) - \eta_n(x_{in}, x_{jn})].$$

Now, a procedure similar to that described in [8] for characteristic values of kernels leads to the inequalities

$$|\widetilde{\mu}_{kn}^2 - \nu_{kn}| \leqslant M_{1n}(\eta_n), \qquad k = 1, 2, \ldots,$$

where $\widetilde{\mu}_{kn} = 0$ for $k > n$, which together with (33) yields

$$|\widetilde{\mu}_{kn}^2 - \widetilde{\mu}_k^2| \leqslant M_1(\eta_n) + M_{1n}(\eta_n), \qquad k = 1, 2, \ldots .$$

Corollary 3 follows from (33).

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