Cubature Formulas of Degree Nine
for Symmetric Planar Regions

By Robert Piessens and Ann Haegemans*

Abstract. A method of constructing 19-point cubature formulas with degree of exactness 9 is given for two-dimensional regions and weight functions which are symmetric in each variable. For some regions, e.g., the square and the circle, these formulas can be reduced to 18-point formulas.

1. Introduction. We consider cubature formulas,

$$(1) \qquad \iint_R w(x, y) f(x, y) \, dx \, dy \cong \sum_{k=1}^{N} w_k f(x_k, y_k),$$

which are exact for all polynomials in x and y of degree $\leq d$ but not for all polynomials of degree $d + 1$. Such formulas are said to have degree d.

According to Rabinowitz and Richter [1], we say that formula (1) is a 'good' formula if it has all of its points (x_k, y_k) inside the region R and all of its coefficients w_k positive. We assume that R is a symmetric region (i.e., $(x, y) \in R$ implies $(\pm x, \pm y) \in R$) and that $w(x, y)$ is symmetric in x and y and nonnegative. In several recent publications [1]–[5], cubature formulas are computed which have the minimum number of points for their degree. Minimum-point formulas are closely connected with the theory of orthogonal polynomials [6]. This theory, however, is not yet sufficiently developed to give practical results in the case of high degree of exactness ($d > 7$). For $9 \leq d \leq 15$, Rabinowitz and Richter [1] have computed perfectly symmetric formulas, by solving a system of nonlinear equations. For $d = 9$, the number of points in their formulas is $N = 20$.

Until now, no formulas of degree 9 with less than 20 points were known for the square; for the circle, only one 19-point formula is computed by Albrecht [7]. In this note, we describe a numerical method for the construction of 18- and 19-point formulas, if they exist.

It is very likely, but not proved, that for the square and the circular domain with weight function $w(x) \equiv 1$ the 18-point formulas constructed in this way are minimum-point formulas.

2. Method of Construction. Since the region and the weight function are symmetric, it is reasonable to consider only symmetric formulas. Firstly, we construct 19-point formulas. As we shall show further, these formulas must have at least 12 points (x_k, y_k) with $x_k \neq 0$ and $y_k \neq 0$. We consider then the formula,

$$\iint_R w(x, y)f(x, y)\,dx\,dy$$

(2)
$$\cong \sum_{k=1}^{3} w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k, -y_k) + f(-x_k, -y_k)]$$

$$+ w_4 [f(x_4, 0) + f(-x_4, 0)] + \sum_{k=5}^{6} w_k [f(0, y_k) + f(0, -y_k)] + w_7 f(0, 0),$$

where $x_k > 0$ and $y_k > 0$.

For the computation of the coefficients w_k and the points (x_k, y_k) of (2), we have the following set of equations

(3.a)
$$4 \sum_{k=1}^{3} w_k x_k^{2\alpha} y_k^{2\beta} = m_{2\alpha, 2\beta}, \quad 0 < \alpha + \beta \leqslant 4,$$

(3.b)
$$4 \sum_{k=1}^{3} w_k x_k^{2\alpha} + 2w_4 x_4^{2\alpha} = m_{2\alpha, 0}, \quad 0 < \alpha \leqslant 4,$$

(3.c)
$$4 \sum_{k=1}^{3} w_k y_k^{2\beta} + 2 \sum_{k=5}^{6} w_k y_k^{2\beta} = m_{0, 2\beta}, \quad 0 < \beta \leqslant 4,$$

(3.d)
$$4 \sum_{k=1}^{3} w_k + 2 \sum_{k=4}^{6} w_k + w_7 = m_{0,0}.$$

Here α and β are natural numbers and

$$m_{\alpha, \beta} = \iint_R w(x, y)x^\alpha y^\beta \, dx\, dy.$$

Firstly, we choose x_2, x_3 and y_3 arbitrarily. The system of six equations (3.a) can then be solved explicitly

$$w_3 = \frac{(m_{2,4}^2 m_{6,2} + m_{4,4}^2 m_{2,2} - 2m_{4,4}m_{2,4}m_{4,2} - m_{6,2}m_{2,2}m_{2,6} + m_{4,2}^2 m_{2,6})}{\{[(m_{2,4}^2 - m_{2,2}m_{2,6})x_3^4 + (m_{4,2}^2 - m_{6,2}m_{2,2})y_3^4}$$

$$+ 2(m_{4,2}m_{2,6} - m_{4,4}m_{2,4})x_3^2 + 2(m_{2,4}m_{6,2} - m_{4,4}m_{4,2})y_3^2$$

$$+ 2(m_{4,4}m_{2,2} - m_{2,4}m_{4,2})x_3^2 y_3^2 + m_{4,4}^2 - m_{6,2}m_{2,6}]4x_3^2 y_3^2\},$$

(4)
$$x_1^2 = (m_{4,2}^* x_2^2 - m_{6,2}^*)/(m_{2,2}^* x_2^2 - m_{4,2}^*),$$

$$y_1^2 = (m_{2,4}^* x_2^2 - m_{4,4}^*)/(m_{2,2}^* x_2^2 - m_{4,2}^*),$$

$$y_2^2 = [(m_{4,4}^* m_{2,2}^* - m_{2,4}^* m_{4,2}^*)x_2^2 + m_{2,4}^* m_{6,2}^* - m_{4,4}^* m_{4,2}^*]/(m_{6,2}^* m_{2,2}^* - m_{4,2}^{*2}),$$

$$w_1 = (m_{2,2}^* x_2^2 - m_{4,2}^*)^2/[4x_1^2 y_1^2(m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)],$$

$$w_2 = (m_{6,2}^* m_{2,2}^* - m_{4,2}^{*2})/[4x_1^2 y_1^2(m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)],$$

where $m_{\alpha,\beta}^* = m_{\alpha,\beta} - 4w_3^2 x_3^\alpha y_3^\alpha$. Formula (4) shows that the coefficient w_3 corresponding to the arbitrarily chosen point (x_3, y_3) is independent of the position of the other points and also different from zero (except perhaps for exceptional regions or weight functions). This means that a symmetric formula of degree 9 requires at least three points in each quadrant of the region.

We consider now the set of four equations (3.b) into which we substitute the free parameters x_2, x_3 and y_3 and the computed values w_1, x_1, y_1, w_2 and y_3. The first two of these equations can be used for the computation of w_4 and x_4. The remaining two equations of this set are then considered as a system of two simultaneous equations in the unknowns x_2, x_3 and y_3. This will generally leave one free parameter, say y_3. The system of nonlinear equations in the unknowns x_2 and x_3 must be solved numerically.

The parameters y_5, y_6, w_5 and w_6 are then computed by solving the system (3.c). The numbers y_5^2 and y_6^2 are the roots of the quadratic equation,

$$(m_{0,4}^{*}{}^2 - m_{0,6}^* m_{0,2}^*)z^2 + (m_{0,2}^* m_{0,8}^* - m_{0,4}^* m_{0,6}^*)z + (m_{0,6}^{*}{}^2 - m_{0,4}^* m_{0,8}^*) = 0,$$

while

$$w_5 = (m_{0,2}^* y_6^2 - m_{0,4}^*)/[2y_5^2(y_6^2 - y_5^2)]$$

and

$$w_6 = (m_{0,4}^* - m_{0,2}^* y_5^2)/[2y_6^2(y_6^2 - y_5^2)]$$

where

$$m_{0,\beta}^* = m_{0,\beta} - 4\sum_{k=1}^{3} w_k y_k^\beta.$$

Finally, w_7 is computed from (3.d).

From this method of solution, we conclude that there are generally more solutions, since y_3 is still a free parameter. However, it is not impossible that several or even all solutions are complex-valued. In this last case, a 19-point formula of the form (2) does not exist. However, there may still exist a formula of the form,

$$\iint_R w(x, y) f(x, y) \, dx \, dy$$

(5)
$$\cong \sum_{k=1}^{4} w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k - y_k) + f(-x_k, -y_k)]$$

$$+ w_5 [f(x_5, 0) + f(-x_5, 0)] + w_6 f(0, 0).$$

In order to obtain an 18-point formula of degree 9, we consider w_7 as a function of y_3 and we solve the equation $w_7(y_3) = 0$ (if there exists a real-valued solution). We conjecture that the 18-point formulas computed in this way are minimum-point formulas.

For several regions and weight functions, we have carried out numerical experiments. We summarize the most important results in the following section.

3. **Some Results.** (i) For the square $R = C_2 = \{(x, y): -1 \leqslant x, y \leqslant 1\}$ and $w(x, y) \equiv 1$ infinitely many 'good' 19-point formulas of degree 9 exist. There are also

at least two 'good' 18-point formulas, the parameters of which we give in Tables 1 and 2 to 20 significant digits. The points of both formulas are common zeros of three orthogonal polynomials of degree 5. These orthogonal polynomials are $P_0 + \lambda_1 P_2$, $P_0 + \lambda_2 P_4$ and $P_1 + \mu_1 P_3 + \mu_2 P_5$ where, for the first formula,

$$\lambda_1 = 0.24819696, \quad \lambda_2 = 0.25574007,$$

$$\mu_1 = 0.00469095, \quad \mu_2 = -0.96079906,$$

and for the second formula,

$$\lambda_1 = -0.22003380, \quad \lambda_2 = -0.30398642,$$

$$\mu_1 = -0.33809909, \quad \mu_2 = -1.33579356,$$

and where

$$P_0(x, y) = x^5 - 10x^3/9 + 5x/21,$$

$$P_1(x, y) = x^4 y - 6x^2 y/7 + 3y/35,$$

$$P_2(x, y) = x^3 y^2 - x^3/3 - 3xy^2/5 + x/5,$$

$$P_3(x, y) = P_2(y, x),$$

$$P_4(x, y) = P_1(y, x),$$

$$P_5(x, y) = P_0(y, x),$$

are the basic orthogonal polynomials for C_2.

(ii) For the circle $R = S_2 = \{(x, y): x^2 + y^2 \leqslant 1\}$ with $w(x, y) \equiv 1$, infinitely many 'good' 19-point formulas of degree 9 exist. We have also computed one 18-point formula which has, however, four of its points outside S_2 (see Table 3). The points of this formula are the common zeros of the orthogonal polynomials $P_0 + \lambda_1 P_2$, $P_0 + \lambda_2 P_4$ and $P_1 + \mu_1 P_3 + \mu_2 P_5$ with

$$\lambda_1 = 0.56685388, \quad \lambda_2 = 0.16924976,$$

$$\mu_1 = 0.56433371, \quad \mu_2 = -0.65980030,$$

and where

$$P_0(x, y) = x^5 - x^3 + 3x/16,$$

$$P_1(x, y) = x^4 y - 3x^2 y/5 + 3y/80,$$

$$P_2(x, y) = x^3 y^2 - x^3/10 - 3xy^2/10 + 3x/80,$$

$$P_3(x, y) = P_2(y, x),$$

$$P_4(x, y) = P_1(y, x),$$

$$P_5(x, y) = P_0(y, x).$$

(iii) For the entire plane $R = E\zeta_2^2 = \{(x, y): -\infty \leqslant x, y \leqslant \infty\}$ with weight function $w(x, y) = \exp(-x^2 - y^2)$, infinitely many 'good' 19-point formulas exist. However, we have not found any 18-point formula.

(iv) For the entire plane $R = E_2^r = \{(x, y): -\infty \leqslant x, y \leqslant \infty\}$ with weight function $w(x, y) = \exp(-(x^2 + y^2)^{1/2})$, we have not found any real solution of the system of equations (3.a, b, c, d).

A number of the 19-point formulas for C_2, S_2 and $E\zeta_2^2$ are tabulated in [8].

TABLE 1. First 18-point formula for the square

k	x_k	y_k	w_k
1	0.8798072139975285389 6	0.9279796150926852886 1	0.6841652246230930567 9 (-1)
2	0.5044591031547983845 6	0.7534719910316150538 0	0.2790338420968730139 5
3	0.9153123540822732418 3	0.4229935709487651306 6	0.1680653382299958712 6
4	0.5788282601192917054 6	0	0.4092735955543314432 9
5	0	0.9770009015800424605 9	0.1064801178156023185 4
6	0	0.3936405727184889351 2	0.4532148810517098563 8

TABLE 2. Second 18-point formula for the square

k	x_k	y_k	w_k
1	0.9374266662206671091 4	0.9414511929992843097 4	0.4285331724889708853 6 (-1)
2	0.5707700168685740441 5	0.7921465451684724753 1	0.2578840636065964430 4
3	0.8977422417984857297 0	0.4000173389763369286 0	0.1939774403700397087 2
4	0.4947178796515962340 9	0	0.4521239831214854997
5	0	0.9808569719466405442 2	0.1024321527099149582 1
6	0	0.4831146961972796564 2	0.4560142235268700112 2

TABLE 3. 18-point formula for the circle

k	x_k	y_k	w_k
1	0.8668687680149229162 2	0.2837667181209480082 7	0.1293726159842295867 0
2	0.6392530693919911468 0	0.9540963986293305456 3	0.7778554090048335511 5 (-2)
3	0.4864519147077642679 6	0.6398245701338767635 9	0.2271330509445306065 1
4	0.5128678920660771865 6	0	0.3209067396138178151 8
5	0	0.8885995350303579785 4	0.1604287073030843962 4
6	0	0.3533551736935300769 0	0.3608924378403773503 6

Applied Mathematics and Programming Division
University of Leuven
Leuven, Belgium

1. P. RABINOWITZ & N. RICHTER, "Perfectly symmetric two-dimensional integration formulas with minimal number of points," *Math. Comp.*, v. 23, 1969, pp. 765–779. MR 41 #2928.

2. I. P. MYSOVSKIH, "On the construction of cubature formulas with fewest nodes," *Dokl. Akad. Nauk SSSR*, v. 178, 1968, pp. 1252–1254 = *Soviet Math. Dokl.*, v. 9, 1968, pp. 277–280. MR 36 #7328.

3. R. FRANKE, "Minimal point cubatures of precision seven for symmetric regions," *SIAM J. Numer. Anal.*, v. 10, 1971, pp. 849–882.

4. S. HABER, "Numerical evaluation of multiple integrals," *SIAM Rev.*, v. 12, 1970, pp. 481–526. MR 44 #2342.

5. A. H. STROUD, *Approximate Calculation of Multiple Integrals*, Prentice-Hall Ser. in Automatic Computation, Prentice-Hall, Englewood Cliffs, N. J., 1971. MR 48 #5348.

6. A. H. STROUD, "Integration formulas and orthogonal polynomials for two variables," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 222–229. MR 41 #6400.

7. J. ALBRECHT, "Formeln zur numerischen Integration über Kreisbereiche," *Z. Angew. Math. Mech.*, v. 40, 1960, pp. 514–517. MR 22 #11514.

8. A. HAEGEMANS & R. PIESSENS, *Tables of Cubature Formulas of Degree Nine for Symmetric Planar Regions.* (Report to be published.)