MATHEMATICS OF COMPUTATION, VOLUME 29, NUMBER 131
JULY 1975, PAGES 834–836

On Lower and Upper Bounds of the Difference Between the Arithmetic and the Geometric Mean

By S. H. Tung

Abstract. Lower and upper bounds of the difference between the arithmetic and the geometric mean of n quantities are given here in terms of n, the smallest value a and the largest value A of given n quantities. Also, an upper bound for the difference, independent of n, is given in terms of a and A. All the bounds obtained are sharp.

1. **Introduction.** Let $a_1, \ldots, a_n$ be n quantities such that $0 < a \equiv a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \equiv A$. Let A_n be their arithmetic mean, G_n their geometric mean. Trivial lower and upper bounds of the difference $A_n - G_n$ are 0 and $A - a$ respectively. A nice upper bound has been obtained in [2]. Here we shall prove the following inequalities:

$$n^{-1}(\sqrt{A} - \sqrt{a})^2 \leqslant A_n - G_n \leqslant ca + (1 - c)A - a^c A^{1-c}$$

where

$$c = \frac{\log[(A/(A - a))\log A/a]}{\log A/a}.$$

The inequalities give lower and upper bounds of the difference $A_n - G_n$ in terms of the smallest value a and the largest value A of the given n quantities. Instead of a discrete method, a continuous and analytic approach is used to obtain the inequalities.

2. **Lower Bounds.** We consider the lower bound of n quantities $0 < a \equiv a_1 \leqslant \cdots \leqslant a_{k-1} \leqslant a_{k+1} \leqslant \cdots \leqslant a_n \equiv A$ with $a_k \equiv x$, $1 < k < n$, to be a variable in the interval $[a, A]$. Let the arithmetic and the geometric means of the fixed $n - 1$ quantities be A_{n-1} and G_{n-1}, respectively. Then

$$A_n - G_n = n^{-1}\{(n - 1)A_{n-1} + x\} - \{G_{n-1}^{n-1}x\}^{1/n} \equiv D_n(x).$$

Since $D'_n(x) = 0$ at $x = G_{n-1}$, the lower bound of $D_n(x)$ for x in the interval $[a, A]$ is

$$D_n(G_{n-1}) = ((n - 1)/n)(A_{n-1} - G_{n-1}).$$

This result can also be found in [1, p. 12], but the method used here seems to be simpler and more straightforward. By repeating this process, we have

$$A_n - G_n \geqslant \frac{n-1}{n}(A_{n-1} - G_{n-1}) \geqslant \frac{n-1}{n}\frac{n-2}{n-1}(A_{n-2} - G_{n-2}) \geqslant \cdots$$

$$\geqslant \frac{2}{n}(A_2 - G_2) = \frac{2}{n}\left(\frac{a + A}{2} - \sqrt{aA}\right) = \frac{1}{n}(\sqrt{A} - \sqrt{a})^2.$$

Received April 18, 1974.
AMS (MOS) subject classifications (1970). Primary 26A86.
Key words and phrases. Arithmetic and geometric mean, inequality, lower and upper bounds.

Equality holds only if $a_2 = \cdots = a_{n-1} = \sqrt{a_1 a_n} = \sqrt{aA}$.

3. Upper Bounds. Now we investigate the upper bound of $A_n - G_n$. The maximum of $D_n(x)$ on $[a, A]$ is attained at the endpoint a or A. Thus, the maximum of $A_n - G_n$ of n quantities is attained when $a \equiv a_1 = \cdots = a_k \leqslant a_{k+1} = \cdots = a_n \equiv A$ for some k, $1 < k < n$, with the form

$$\frac{ka + (n-k)A}{n} - \{a^k A^{n-k}\}^{1/n} = a\left[\frac{k + (n-k)A/a}{n} - \{A/a\}^{(n-k)/n}\right].$$

For the sake of simplicity, let $a = 1$ and consider the function

$$D(x) = \frac{x + (n-x)A}{n} - A^{(n-x)/n}.$$

Through straight calculation, we have $D'(x) = 0$ for $x = cn$, where

$$c = \frac{\log[(A/(A-1))\log A]}{\log A}.$$

Thus, the upper bound for $A_n - G_n$ is $D(cn) = c + (1-c)A - A^{1-c} \equiv U$ which is independent of n. By repeated application of L'Hospital's rule, we have

$$\lim_{A \to 1} c = \frac{1}{2} \quad \text{and} \quad \lim_{A \to \infty} c = 0.$$

The upper bound is attained only at its limiting case $A \longrightarrow 1$ or $n \longrightarrow \infty$. But this is the best possible bound independent of the positive integer $n \geqslant 2$. For a fixed n, the sharp upper bound of $A_n - G_n$ is attained by $D(k_n)$ with $k_n = [cn]$ or $[cn] + 1$, where $[cn]$ denotes the largest integer not greater than cn. Therefore, we have lower and upper bounds of $A_n - G_n$, both dependent and independent of n, in terms of $a = 1$ and A as follows:

$$0 \leqslant n^{-1}(\sqrt{A} - 1)^2 \leqslant A_n - G_n \leqslant D(k_n) \leqslant c + (1-c)A - A^{1-c}.$$

Some numerical data are shown below.

TABLE ($a = 1$)

A	c	k_{10}	$D(k_{10})$	U
1	0.5		0	0
1.001	0.499925	5	1.25×10^{-7}	1.25×10^{-7}
1.1	0.496029	5	1.191152×10^{-3}	1.191227×10^{-3}
2	0.471234	5	0.085786	0.086071
e	0.458675	5	0.210420	0.211867
5	0.434331	4	0.773472	0.777337
10	0.407973	4	2.418928	2.419591
100	0.333805	3	45.18114	45.45570
10^5	0.212238	2	7.00002×10^4	7.00906×10^4
10^{10}	0.136222	1	8.00000×10^9	8.20349×10^9

Department of Mathematics and Statistics
Miami University
Oxford, Ohio 45056

1. E. F. BECKENBACH & R. E. BELLMAN, *Inequalities,* 2nd rev. ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 30, Springer-Verlag, New York, 1965. MR 33 #236.

2. C. LOEWNER & H. B. MANN, "On the difference between the geometric and the arithmetic mean of n quantities," *Advances in Math.,* v. 5, 1971, pp. 472–473. MR 43 #4982.