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On the Distribution of the Zeros of Generalized Airy Functions

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Abstract. We give tables of zeros and values of the generalized Airy functions introduced by Swanson and Headley [*SIAM J. Appl. Math.*, v. 15, 1967, pp. 1400–1412]. The tables enable us to sharpen substantially results on the distribution of the zeros. We show that the nonreal zeros are asymptotically close to the boundary rays of the sectors obtained in the paper cited. We conjecture from the numerical evidence that the zeros monotonically approach the rays.

1. **Introduction.** In a previous article [9], Swanson and one of the present authors discussed a pair of linearly independent solutions of the ordinary differential equation

$$d^2u/dz^2 - z^n u = 0, \tag{1}$$

where z is a complex variable and n is a positive integer. This differential equation is of interest because, *inter alia,* it is the simplest ordinary differential equation of the second order with a turning point of order n. It is thus a natural comparison equation to use when seeking uniform asymptotic expansions in unbounded regions containing a turning point of order n for general second-order equations. The equation we are discussing also arises in various physical situations; for example, the propagation of waves in varying media [1].

In [9] two linearly independent solutions $A_n(z)$, $B_n(z)$ were chosen in such a manner that they became, for $n = 1$, the well-known Airy functions discussed in detail by Miller [3], Olver [6], Swanson [8], and others. It was shown [9] that the zeros of $A_n(z)$, $B_n(z)$ and their derivatives were restricted to certain sectors of the complex z-plane. Since it is known [6] that the real zeros of $A_1(z)$ and $B_1(z)$ lie along the negative real axis, we asked ourselves whether we could obtain more precise information regarding the asymptotic distribution of the large nonreal zeros of $A_n(z)$, $B_n(z)$ and their derivatives. This information, if obtainable, would permit much sharper estimates of the maximum domain of validity for the uniform asymptotic expansions mentioned above. We, therefore, wrote a computer program to calculate the zeros, using reversion of the asymptotic series for the solutions of (1). (The tables, which also include the zeros of the derivatives $A'_n(z)$, $B'_n(z)$ and the values of $A_n(z)$, $B_n(z)$ at the zeros of $A'_n(z)$, $B'_n(z)$, appear at the end of this paper.) By inspecting these results, we were led to conjecture and prove that the large zeros z_m lie close to the farther boundary ray of each of the sectors discovered in [9]; indeed there is empirical evidence that in each of these sectors, $|\arg(z_m)|$ increases strictly with m. Inspection of the tables also reveals

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that even the small zeros lie close to the appropriate boundary ray. For example, if we consider $A_n(z)$, we see that $\arg(z_1)$ is within 0.04 radians of the boundary so that more than eighty percent of the sector is in fact zero-free. The results obtained in the present paper therefore constitute a substantial sharpening of known results, even in the classical case $n = 1$.

To improve the accuracy of the small zeros, we used Aitken's method with ascending series, then the results were checked by comparing them with known results [3] for $n = 1$ and by substituting into the ascending series and the asymptotic series for the solutions of (1).

2. Definitions and Background Material. As in [9], we define the functions A_n and B_n by the relations

(2) $$A_n(z) = pz^{1/2}[I_{-p}(\zeta) - I_p(\zeta)],$$

(3) $$B_n(z) = (pz)^{1/2}[I_{-p}(\zeta) + I_p(\zeta)],$$

where

(4) $$\zeta = 2pz^{1/(2p)}, \qquad p = 1/(n + 2),$$

and I_p denotes the modified Bessel function of the first kind of order p. In the case $n = 1$, the functions A_n and B_n reduce to the usual Airy functions Ai and Bi, respectively.

It follows from the standard theory of Bessel functions [11] that $A_n(z)$ and $B_n(z)$ are real on the real axis, analytic in the finite complex plane, and are linearly independent solutions of the differential equation (1).

According to [9], the functions A_n and A'_n have no real zeros for even n; but for odd n, they have an infinite sequence of simple zeros on the negative real axis with no finite accumulation point. Moreover, nonreal zeros of A_n and A'_n are all contained in the region $\bigcup_k S_k$, where k takes all integer values in the range $0 < 2k < n + 1$, and S_k is the double sector

$$\{z: 2k\pi/(n + 1) < |\arg z| < (2k + 1)\pi/(n + 2)\}.$$

An analogous result holds for B_n and B'_n; but in this case, the nonreal zeros lie in the region $\bigcup_k S'_k$, where k takes all integer values in the range $0 < 2k < n + 3$, and S'_k is the double sector

$$\{z: (2k - 1)\pi/(n + 2) < |\arg z| < (2k - 1)\pi/(n + 1)\}.$$

3. Asymptotic Distribution of the Zeros.

THEOREM 1. *For each positive integer n, the zeros of A_n or A'_n which lie in the double sector S_k $(0 < 2k < n + 1)$ approach one of the rays $|\arg z| = (2k + 1)\pi/(n + 2)$. That is, if the zeros in the sector S_k are arranged in ascending order of absolute value: $|z_{k1}| < |z_{k2}| < \dots < |z_{km}| < \dots$, then*

$$\lim_{m\to\infty} |\arg z_{km}| = \frac{(2k + 1)\pi}{n + 2} \qquad \left(k = 1, 2, \dots, \left[\frac{n + 1}{2}\right]\right).$$

Moreover, the distance between successive zeros approaches zero as m approaches infinity.

Proof. For a zero z of A_n or A_n' in the upper half-plane, let $\zeta = Ze^{(k+1/2)\pi i}$. Then [11, pp. 77–78]

$$(5)\qquad I_p(\zeta) = e^{(k+1)p\pi i} I_p(Ze^{-i\pi/2}) = e^{(k+1/2)p\pi i} J_p(Z) \qquad (-\pi/2 < \arg Z \leqslant \pi),$$

and

$$(6)\qquad I_{-p}(\zeta) = e^{-(k+1/2)p\pi i} J_{-p}(Z) \qquad (-\pi/2 < \arg Z \leqslant \pi).$$

Making use of the relation [11, p. 74]

$$J_{-p}(Z) = J_p(Z)\cos p\pi - Y_p(Z)\sin p\pi,$$

where Y_p is Weber's Bessel function of the second kind of order p, we see that, for $-\pi/2 < \arg Z \leqslant \pi$,

$$I_{-p}(\zeta) - I_p(\zeta) = [e^{-(k+1/2)p\pi i}\cos p\pi - e^{(k+1/2)p\pi i}] J_p(Z)$$
$$- e^{-(k+1/2)p\pi i} Y_p(Z)\sin p\pi.$$

It follows that there exists a complex number α such that

$$(7)\qquad I_{-p}(\zeta) - I_p(\zeta) = J_p(Z)\cos\alpha - Y_p(Z)\sin\alpha.$$

To see this, we note that the equation $\tan\alpha = \omega$ has solutions given by

$$(8)\qquad e^{i\alpha} = \pm\sqrt{(1 + i\omega)/(1 - i\omega)};$$

hence α is well defined when $\omega \neq \pm i$. If we now set

$$(9)\qquad \tan\alpha = \omega = \frac{e^{-(k+1/2)p\pi i}\sin p\pi}{e^{-(k+1/2)p\pi i}\cos p\pi - e^{(k+1/2)p\pi i}} = \frac{\sin p\pi}{\cos p\pi - e^{(2k+1)p\pi i}},$$

we see that the exceptional values $\pm i$ of ω occur when

$$\cos p\pi = \cos(2k+1)p\pi \quad \text{and} \quad \sin p\pi = \pm\sin(2k+1)p\pi;$$

that is, when $(2k + 1)p\pi = 2l\pi \pm p\pi$ $(l = 1, 2, 3, \ldots)$. But $p = 2/(n + 2)$, hence the exceptional values of ω only occur when $2k + 1 = l(n + 2)$ $(l = 1, 2, 3, \ldots)$. Since $0 < 2k < n + 1$, it follows that the exceptional values of ω cannot occur. Thus, a complex number α can always be found satisfying (7). It is known [11, pp. 505–506] that the large zeros Z_{km} of $J_p(Z)\cos\alpha - Y_p(Z)\sin\alpha$ have the asymptotic expansion

$$(10)\qquad Z_{km} \sim b_{km} - \frac{4p^2 - 1}{8b_{km}} - \frac{(4p^2 - 1)(28p^2 - 31)}{384 b_{km}^3} - \cdots, \qquad (m \longrightarrow \infty),$$

where $b_{km} = (m + \tfrac{1}{2}p - \tfrac{1}{4})\pi - \alpha$. It is clear from (9) that α is independent of m. Let $\alpha = \alpha_1 + i\alpha_2$. Then it follows from (10) that $\mathrm{Im}(Z_{km}) = -\alpha_2 + O(b_{km}^{-1})$ $(m \longrightarrow +\infty)$,

$$\operatorname{Re}(Z_{km}) = (m + \tfrac{1}{2}p - \tfrac{1}{4})\pi - \alpha_1 + O(b_{km}^{-1}) \qquad (m \longrightarrow +\infty),$$

hence $\lim_{m\to\infty} \operatorname{Im}(Z_{km})/\operatorname{Re}(Z_{km}) = 0$. The corresponding zero z_{km} of $A_n(z)$ satisfies the relation

$$\arg z_{km} = \frac{n+2}{2} \arg Z_{km} + \frac{(2k+1)\pi}{n+2}.$$

It follows that $\lim_{m\to\infty} \arg z_{km} = (2k+1)/(n+2)$. With k fixed, abbreviate Z_{km}, z_{km}, b_{km} by Z_m, z_m, b_m, respectively. Let $Z_{m+1} - Z_m = \epsilon_m$. Then $\epsilon_m = \pi + O(b_m^{-1})$ $(m \longrightarrow \infty)$.

By (4) and the definition of Z we must have

$$z_m = [Z_m/(2p)]^{2p} e^{(2k+1)p\pi i}$$

and

$$z_{m+1} = [Z_{m+1}/(2p)]^{2p} e^{(2k+1)p\pi i}.$$

Let

$$t_m = \left(\frac{Z_m + \epsilon_m}{2p}\right)^{2p} - \left(\frac{Z_m}{2p}\right)^{2p} \qquad (m = 1, 2, 3, \ldots).$$

Then

$$\begin{aligned} t_m &= \left(\frac{Z_m}{2p}\right)^{2p} \left[\left(1 + \frac{\epsilon_m}{Z_m}\right)^{2p} - 1\right] \\ &= \left(\frac{Z_m}{2p}\right)^{2p} \left[\frac{2p\epsilon_m}{Z_m} + \frac{2p(2p-1)\epsilon_m^2}{2Z_m^2} + O\left(\frac{\epsilon_m}{Z_m}\right)^3\right] \\ &= \left(\frac{2p}{Z_m}\right)^{1-2p} \epsilon_m + O\left(\frac{1}{Z_m}\right)^{2-2p} \qquad (m \longrightarrow \infty). \end{aligned}$$

It follows that $\lim_{m\to\infty} t_m = 0$; and, therefore, $\lim_{m\to\infty}(z_{m+1} - z_m) = 0$. Since the zeros are symmetrically placed with respect to the real axis, the proof of the theorem is now complete.

THEOREM 2. *For each positive integer n, the zeros of B_n or B_n' which lie in the double sector in S_k' $(0 < 2k < n+3)$, approach one of the rays* $|\arg z| = (2k-1)\pi/(n+2)$. *Moreover, if z_m and z_{m+1} are successive zeros in S_k', with $|z_m| < |z_{m+1}|$ $(m = 1, 2, 3, \ldots)$, then*

$$\lim_{m\to\infty} |z_{m+1} - z_m| = 0.$$

The proof will be omitted since it is similar to that of Theorem 1.

REMARK. We conjecture from the numerical and asymptotic evidence that if z_m is a zero of A_n or A_n' in the sector S_k $(0 < 2k < n+1)$, then $|\arg z_m|$ increases strictly with m; and that if z_m is a zero of B_n or B_n' in the sector S_k' $(0 < 2k < n+3)$, then $|\arg z_m|$ decreases strictly with m.

4. Computation of the Zeros. We shall find approximations to the zeros of $A_n(z)$ by reverting the asymptotic series for the general cylinder function given by (7). Accordingly, let ζ_1 be a zero of the cylinder function

$$J_p(\zeta_1) \cos \alpha - Y_p(\zeta_1) \cos \alpha, \tag{11}$$

where

$$p = 1/(n+2), \qquad \zeta_1 = 2pz^{1/(2p)}e^{-(k+1/2)\pi i}, \tag{12}$$

$$\tan \alpha = \frac{\sin p\pi}{\cos p\pi - \exp[(2k+1)p\pi i]}, \tag{13}$$

and k is an integer in the range $0 < 2k < n + 1$. It follows from known results [11, pp. 504–506] that the rth zero of (11) is given by

$$\psi = \zeta_1 - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi + \alpha - r\pi, \tag{14}$$

where (14) is obtained by integrating the asymptotic series

$$1 - \frac{d\psi}{d\zeta_1} \sim \left[\sum_{m=0}^{\infty} \{1 \cdot 3 \cdot 5 \ldots (2m-1)\} \frac{(p, m)}{2^m \zeta_1^{2m}}\right]^{-1} \tag{15}$$

with

$$(p, 0) = 1, \qquad (p, m+1) = \left[\frac{4p^2 - (2m+1)^2}{2^2(m+1)}\right](p, m) \qquad (m = 0, 1, 2, \ldots).$$

It should be noted that if z is a zero of $A_n(z)$ in the double sector $\{z: 2k\pi/(n+1) < |\arg z| < (2k+1)\pi/(n+2)\}$, then it follows from (12) that $|\arg \zeta_1| < \pi$; and, therefore, the asymptotic expansion (15) is applicable [11, p. 449]. On integrating (15) and making use of (14), we obtain the asymptotic representation

$$\zeta_1 - (k + \tfrac{1}{2}p - \tfrac{1}{4})\pi + \alpha \sim \sum_{m=1}^{\infty} \frac{b_{m+1}}{(2m-1)\zeta_1^{2m-1}}, \tag{16}$$

where

$$b_1 = 1, \qquad b_m = -\sum_{j=2}^{m} a_j b_{m+1-j} \qquad (m = 2, 3, 4, \ldots),$$

$$a_1 = 1, \qquad a_{m+1} = \frac{(2m-1)[4p^2 - (2m-1)^2]}{2^3 m} a_m \qquad (m = 0, 1, 2, \ldots).$$

Finally, we set $\beta = (k + \tfrac{1}{2}p - \tfrac{1}{4})\pi - \alpha$ and expand z_1 as a descending series in β. It turns out that if we go as far as the term in β^{-9}, we obtain results accurate to eight decimal places for $r \geqslant 4$.

To improve the accuracy of the first three zeros in each sector, we start with the ascending series obtained in [9], writing it in the form

$$t = p\Gamma(p) \sum_{m=0}^{\infty} \frac{t^{m(n+2)}}{m!(m+1-p)} - \sum_{m=1}^{\infty} \frac{t^{m(n+2)+1}}{m!\Gamma(m+1+p)}, \tag{17}$$

where $t = p^{2p}z$ and z is a zero of $A_n(z)$ in the upper half of the double sector S_k. We now perform the rotation

$$t_1 = t/w, \qquad w = \exp[(2k+1)p\pi], \tag{18}$$

so as to bring the zero as close as possible to the positive real axis. Finally, we use (17) in the form

$$t_1 \approx \frac{1}{w} p\Gamma(p) \sum_{m=0}^{L} \frac{t^{m(n+2)}}{m!\Gamma(m+1-p)} - \sum_{m=1}^{L} \frac{t^{m(n+2)+1}}{m!\Gamma(m+1+p)} \tag{19}$$

to generate successive approximations to t_1 for $L = 0, 1, 2, \ldots$, each iterate of t_1 being sharpened by Aitken's Δ^2-method before substitution into (19) for the next approximation.

The zeros of $A'_n(z)$, $B_n(z)$, $B'_n(z)$ were computed similarly by making suitable changes in (11)–(19).

The tables at the end of the paper were prepared from computer printout, hence a slight change in notation has been made. Instead of $A_N(Z)$, $B_N(Z)$, $A'_N(Z)$, $B'_N(Z)$, the tables have $AN(Z)$, $BN(Z)$, $AN'(Z)$, $BN'(Z)$. The zeros of $AN(Z)$ and $AN'(Z)$ lie in the double sectors

$$\{Z\colon 2I\pi/(N+1) < |\arg(Z)| < (2I+1)\pi/(N+2)\},$$

where I takes all integer values in the range $0 < 2I < N + 1$. The zeros of $BN(Z)$ and $BN'(Z)$ lie in the double sectors

$$\{Z\colon (2I-1)\pi/(N+2) < |\arg(Z)| < (2I-1)\pi/(N+1)\},$$

where I takes all integer values in the range $0 < 2I < N + 3$. In each sector, the zeros are listed in ascending order of their absolute value. We give only the first ten zeros in each sector, but we note that the technique used gives accurate results for all the zeros.

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TABLE 1

			ZEROS OF AN(Z)		AN'(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
1	1	1	2.33810741	3.14159265	0.70121082	0.0
		2	4.08794944	3.14159265	0.80311137	3.14159265
		3	5.52055983	3.14159265	0.86520403	0.0
		4	6.78670809	3.14159265	0.91085074	3.14159265
		5	7.94413359	3.14159265	0.94733571	0.0
		6	9.02265086	3.14159265	0.97792281	3.14159265
		7	10.04017434	3.14159265	1.00437012	0.0
		8	11.00852430	3.14159265	1.02773869	3.14159265
		9	11.93601556	3.14159265	1.04872065	0.0
		10	12.82877675	3.14159265	1.06779386	3.14159265
2	1	1	2.19034316	2.32053093	0.70687774	0.37569934
		2	3.32181119	2.34053384	0.86594896	-2.75663573
		3	4.15977716	2.34619170	0.96820857	0.38772177
		4	4.85595843	2.34885034	1.04580284	-2.75255596
		5	5.46440016	2.35039341	1.10925011	0.38980335
		6	6.01168328	2.35140096	1.16339621	-2.75128761
		7	6.51319779	2.35211043	1.21090419	0.39065874
		8	6.97879729	2.35263703	1.25304069	-2.75067119
		9	7.41523962	2.35304338	1.29198370	0.39112430
		10	7.82739765	2.35336642	1.32738810	-2.75030704
3	1	1	2.05058948	1.84553066	0.65135310	0.60002539
		2	2.85762476	1.86757748	0.83061108	-2.52617108
		3	3.42040924	1.87384927	0.94959617	0.62002617
		4	3.87092999	1.87679966	1.04161221	-2.51937602
		5	4.25419681	1.87851274	1.11788529	0.62349387
		6	4.59170790	1.87963151	1.18367563	-2.51726294
		7	4.89563611	1.88041940	1.24190933	0.62491901
		8	5.17363567	1.88100423	1.29439784	-2.51623591
		9	5.43087666	1.88145552	1.34234796	0.62569469
		10	5.67103832	1.88181432	1.38660815	-2.51562920
4	1	1	1.93480106	1.53339438	0.59108461	0.74952439
		2	2.54957321	1.55427477	0.77419136	-2.37255136
		3	2.96133373	1.56023389	0.89832191	0.77488084
		4	3.28288202	1.56303894	0.99553466	-2.36393369
		5	3.55155083	1.56466802	1.07684986	0.77927890

TABLE 1 (*Continued*)

			ZEROS OF AN(Z)		AN'(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
4	1	6	3.78482274	1.56573204	1.14748965	-2.36125365
		7	3.99244987	1.56648141	1.21038250	0.78108643
		8	4.18049055	1.56703768	1.26735289	-2.35995103
		9	4.35299545	1.56746694	1.31962279	0.78207025
		10	4.51282190	1.56780823	1.36805515	-2.35918153
4	2	1	1.93112085	2.60816427	0.83449806	0.25237644
		2	2.54860724	2.61366431	1.09446892	-2.88407953
		3	2.96087343	2.61522715	1.27022404	0.25904449
		4	3.28260643	2.61596220	1.40778125	-2.88182018
		5	3.55136466	2.61638896	1.52281620	0.26019683
		6	3.78468721	2.61666765	1.62273749	-2.88111815
		7	3.99234607	2.61686392	1.71169494	0.26067026
		8	4.18040807	2.61700960	1.79227234	-2.88077699
		9	4.35292805	2.61712202	1.86619960	0.26092791
		10	4.51276562	2.61721140	1.93469805	-2.88057547
5	1	1	1.84076804	1.31208620	0.53687755	0.85639356
		2	2.33112483	1.33122107	0.71677225	-2.26278230
		3	2.65017756	1.33669279	0.84055893	0.88551777
		4	2.89494587	1.33926942	0.93837590	-2.25288391
		5	3.09684672	1.34076604	1.02072758	0.89056939
		6	3.27038435	1.34174361	1.09263147	-2.24980564
		7	3.42356311	1.34243212	1.15691753	0.89264551
		8	3.56131261	1.34294322	1.21535689	-2.24830946
		9	3.68690490	1.34333764	1.26914026	0.89377550
		10	3.80263359	1.34365122	1.31911122	-2.24742562
5	2	1	1.83759911	2.23108481	0.80322026	0.43330355
		2	2.33032279	2.23830188	1.07398710	-2.69984121
		3	2.64980327	2.24035617	1.25977208	0.44426948
		4	2.89472502	2.24132269	1.40648555	-2.69612623
		5	3.09669918	2.24188391	1.52997263	0.44616415
		6	3.27027791	2.24225044	1.63778034	-2.69497197
		7	3.42348220	2.24250857	1.73416001	0.44694253
		8	3.56124874	2.24270017	1.82177053	-2.69441106
		9	3.68685302	2.24284803	1.90239877	0.44736615
		10	3.80259048	2.24296558	1.97731031	-2.69407973

TABLE 2

			ZEROS OF AN'(Z)		AN(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
1	1	1	1.01879297	3.14159265	0.53565666	0.0
		2	3.24819758	3.14159265	0.41901548	3.14159265
		3	4.82009921	3.14159265	0.38040647	0.0
		4	6.16330736	3.14159265	0.35790794	3.14159265
		5	7.37217726	3.14159265	0.34230124	0.0
		6	8.48848673	3.14159265	0.33047623	3.14159265
		7	9.53544905	3.14159265	0.32102229	0.0
		8	10.52766040	3.14159265	0.31318539	3.14159265
		9	11.47505663	3.14159265	0.30651729	0.0
		10	12.38478837	3.14159265	0.30073083	3.14159265
2	1	1	1.14394586	2.19693240	0.41062389	-0.34465562
		2	2.78994738	2.33371004	0.28268036	2.75973248
		3	3.75466841	2.34386350	0.24446498	-0.38660739
		4	4.51583865	2.34768401	0.22308801	2.75312385
		5	5.16550405	2.34969423	0.20865176	-0.38946020
		6	5.74191810	2.35093543	0.19793147	2.75151711
		7	6.26542401	2.35177832	0.18949798	-0.39049455
		8	6.74838513	2.35238821	0.18260067	2.75079443
		9	7.19898515	2.35285002	0.17679969	-0.39102840
		10	7.62297506	2.35321186	0.17181663	2.75038379
3	1	1	1.21759802	1.69477672	0.29735001	-0.54492817
		2	2.48297207	1.85987246	0.18961006	2.53138090
		3	3.15018496	1.87123995	0.15925753	-0.61816012
		4	3.65185588	1.87549610	0.14269055	2.52032542
		5	4.06658703	1.87773243	0.13168124	-0.62292067
		6	4.42582042	1.87911244	0.12360390	2.51764611
		7	4.74584133	1.88004930	0.11731083	-0.62464498
		8	5.03634639	1.88072707	0.11220557	2.51644156
		9	5.30364706	1.88124022	0.10794155	-0.62553469
		10	5.55211563	1.88164226	0.10430103	2.51575722
4	1	1	1.25505627	1.38107258	0.21927675	-0.67640886
		2	2.26657314	1.54686212	0.13283166	2.37920444
		3	2.76452759	1.55773665	0.10941121	-0.77250581
		4	3.12701254	1.56179376	0.09683709	2.36514064
		5	3.42034992	1.56392339	0.08857070	-0.77855062

TABLE 2 (*Continued*)

			ZEROS OF AN'(Z)		AN(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
4	1	6	3.67038911	1.56523700	0.08255454	2.36174030
		7	3.89028283	1.56612861	0.07789754	-0.78073847
		8	4.08775570	1.56677355	0.07413990	2.36021212
		9	4.26777933	1.56726181	0.07101598	-0.78186714
		10	4.43376503	1.56764432	0.06835968	2.35934401
4	2	1	1.16911645	2.55675445	0.31636516	-0.23148051
		2	2.26471167	2.61171567	0.18799041	2.88582741
		3	2.76386315	2.61457241	0.15476664	-0.25842185
		4	3.12665697	2.61563593	0.13696361	2.88213642
		5	3.42012363	2.61619390	0.12526609	-0.26000606
		6	3.67023036	2.61653799	0.11675476	2.88124561
		7	3.89016427	2.61677152	0.11016710	-0.26057913
		8	4.08766320	2.61694043	0.10485201	2.88084537
		9	4.26770478	2.61706830	0.10043351	-0.26087472
		10	4.43370345	2.61716847	0.09667652	2.88061802
5	1	1	1.27242116	1.16595921	0.16638281	-0.76947374
		2	2.10676942	1.32435314	0.09713985	2.27046306
		3	2.49818276	1.33438802	0.07890160	-0.88278284
		4	2.77656148	1.33812180	0.06922535	2.25427255
		5	2.99841912	1.34008025	0.06291299	-0.88973185
		6	3.18537085	1.34128789	0.05834539	2.25036515
		7	3.34827519	1.34210743	0.05482602	-0.89224551
		8	3.49344679	1.34270019	0.05199725	2.24860955
		9	3.62491713	1.34314892	0.04965334	-0.89354208
		10	3.74543360	1.34350045	0.04766608	2.24761235
5	2	1	1.18706890	2.15706794	0.25556171	-0.39712288
		2	2.10518574	2.23571679	0.14573409	2.70273029
		3	2.49763577	2.23949112	0.11830408	-0.44324308
		4	2.77627419	2.24089225	0.10378147	2.69664706
		5	2.99823868	2.24162675	0.09431314	-0.44585010
		6	3.18524557	2.24207958	0.08746360	2.69518175
		7	3.34818241	2.24238684	0.08218665	-0.44679258
		8	3.49337491	2.24260906	0.07794551	2.69452356
		9	3.62485957	2.24277728	0.07443148	-0.44727864
		10	3.74538631	2.24290906	0.07145224	2.69414973

TABLE 3

			ZEROS OF BN(Z)		BN'(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
1	1	1	2.35387338	1.14253251	0.99310685	2.64060025
		2	4.09328731	1.08898311	1.13612833	-0.51328287
		3	5.52350350	1.07387810	1.22374379	2.62462836
		4	6.78865953	1.06678453	1.28822926	-0.51871638
		5	7.94555902	1.06266837	1.33979477	2.62185446
		6	9.02375637	1.05998103	1.38303390	-0.52040694
		7	10.04106737	1.05808881	1.42042535	2.62071419
		8	11.00926726	1.05668440	1.45346646	-0.52122872
		9	11.93664761	1.05560073	1.48313457	2.62009352
		10	12.82932394	1.05473921	1.51010464	-0.52171419
2	1	1	2.19034316	0.82106172	1.41375548	2.76589331
		2	3.32181119	0.80105881	1.73189792	-0.38495692
		3	4.15977718	0.79540096	1.93641714	2.75387089
		4	4.85595843	0.79274231	2.09160568	-0.38903670
		5	5.46440016	0.79119924	2.21850023	2.75178930
		6	6.01168328	0.79019170	2.32679242	-0.39030504
		7	6.51319779	0.78948222	2.42180838	2.75093391
		8	6.97879729	0.78895562	2.50681386	-0.39092147
		9	7.41523962	0.78854928	2.58396740	2.75046835
		10	7.82739765	0.78822623	2.65477619	-0.39128561
3	1	1	2.04746194	0.64572426	1.75035475	2.83991982
		2	2.85676075	0.63597601	2.33384735	-0.30847649
		3	3.41998530	0.63321098	2.55417371	2.83108624
		4	3.87067094	0.63191097	2.80179104	-0.31147157
		5	4.25401905	0.63115629	3.00701098	2.82955841
		6	4.59157685	0.63066347	3.18401245	-0.31240243
		7	4.89553466	0.63031643	3.34067692	2.82893066
		8	5.17355431	0.63005884	4.48188149	-0.31285480
		9	5.43080965	0.62986006	3.61087469	2.82858900
		10	5.67098195	0.62970203	3.72993990	-0.31312202
4	1	1	1.93112085	0.53342839	2.0440945	2.88921621
		2	2.54860724	0.52792834	2.68089039	-0.25751312
		3	2.96087343	0.52636551	3.11140074	2.88254816
		4	3.28260643	0.52563046	3.44834572	-0.25977247
		5	3.55136466	0.52520370	3.73012266	2.88139582

TABLE 3 (*Continued*)

			ZEROS OF BN(Z)		BN'(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
4	1	6	3.78468721	0.52492500	3.97487884	-0.26047450
		7	3.99234607	0.52472873	4.19277919	2.88092239
		8	4.18040807	0.52458305	4.39015272	-0.26081566
		9	4.35292805	0.52447063	4.57123677	2.88066474
		10	4.51276562	0.52438126	4.73902302	-0.26101718
4	2	1	1.93480107	1.60819828	1.44785568	2.39206826
		2	2.54957321	1.58731788	1.89637381	-0.76904130
		3	2.96133373	1.58135876	2.20043031	2.36671181
		4	3.28288202	1.57855369	2.43855193	-0.77765897
		5	3.55155083	1.57692461	2.63773270	2.36231376
		6	3.78482274	1.57586059	2.81076413	-0.78033900
		7	3.99244987	1.57511122	2.96481952	2.36050622
		8	4.18049055	1.57455496	3.10436791	-0.78164162
		9	4.35299545	1.57412570	3.23240250	2.35952241
		10	4.51282190	1.57378442	3.35103707	-0.78241113
5	1	1	1.83719462	0.45490143	2.30987514	2.92451728
		2	2.33022083	0.45148891	3.08913616	-0.22106945
		3	2.64975572	0.45051812	3.62362660	2.91933328
		4	2.89469697	0.45006143	4.04567630	-0.22282494
		5	3.09668044	0.44979626	4.40089972	2.91843805
		6	3.27026439	0.44962309	4.71101512	-0.22337031
		7	3.42347193	0.44950113	4.98825473	2.91807028
		8	3.56124063	0.44941060	5.24026829	-0.22363533
		9	3.68684643	0.44934074	5.47219632	2.91787013
		10	3.80258501	0.44928520	5.68767972	-0.22379187
5	2	1	1.83852512	1.36790487	1.82539101	2.49421373
		2	2.33055656	1.35588960	2.43965146	-0.66144673
		3	2.64991232	1.35246493	2.86147404	2.47594806
		4	2.89478936	1.35085326	3.19464804	-0.66764051
		5	3.09674215	1.34991733	3.47509693	2.47278858
		6	3.27030891	1.34930607	3.71994522	-0.66956545
		7	3.42350577	1.34887557	3.93884286	2.47149043
		8	3.56126734	1.34855601	4.13782639	-0.67050093
		9	3.68686813	1.34830942	4.32095296	2.47078393
		10	3.80260304	1.34811336	4.49109644	-0.67105351

TABLE 4

			ZEROS OF BN'(Z)		BN(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
1	1	1	1.12139329	1.37792417	0.75004149	0.46597789
		2	3.25690823	1.10658749	0.59221663	-2.63235403
		3	4.82400261	1.07998319	0.53787063	0.51549330
		4	6.16568666	1.06986001	0.50611020	-2.62362859
		5	7.37383799	1.06451720	0.48406006	0.51928282
		6	8.48973856	1.06121408	0.46734684	-2.62149057
		7	9.53644071	1.05896948	0.45398232	0.52066021
		8	10.52847375	1.05734473	0.44290250	-2.62052784
		9	11.47574112	1.05611417	0.43347447	0.52137155
		10	12.38537593	1.05514984	0.42529258	-2.61998058
2	1	1	1.14394586	0.94466026	0.82124778	0.34465562
		2	2.78994738	0.80788261	0.56536073	-2.75973248
		3	3.75466841	0.79772916	0.48892996	0.38660739
		4	4.51583865	0.79390865	0.44617601	-2.75312385
		5	5.16550405	0.79189845	0.41730352	0.38946020
		6	5.74191810	0.79065722	0.39586294	-2.75151711
		7	6.26542401	0.78981433	0.37899595	0.39049455
		8	6.74838513	0.78920444	0.36520134	-2.75079443
		9	7.19898515	0.78874263	0.35359938	0.39102840
		10	7.62297506	0.78838079	0.34363326	-2.75038379
3	1	1	1.16184363	0.72171921	0.80837790	0.27622099
		2	2.48136268	0.63937810	0.51027252	-2.83541560
		3	3.14958342	0.63436113	0.42846095	0.30968393
		4	3.65152544	0.63248531	0.38386230	-2.83053938
		5	4.06637280	0.63150004	0.35423559	0.31178173
		6	4.42566796	0.63089212	0.33250229	-2.82935901
		7	4.74572612	0.63047945	0.31557117	0.31254129
		8	5.03625559	0.63018091	0.30183638	-2.82882843
		9	5.30357325	0.62995489	0.29036516	0.31293318
		10	5.55205419	0.62977782	0.28057148	-2.82852701
4	1	1	1.16911645	0.58483820	0.77493322	0.23148051
		2	2.26471167	0.52987698	0.46048057	-2.88582741
		3	2.76386315	0.52702025	0.37909929	0.25842185
		4	3.12665697	0.52595672	0.33549096	-2.88213642
		5	3.42012363	0.52539876	0.30683799	0.26000606

TABLE 4 (*Continued*)

			ZEROS OF BN'(Z)		BN(Z) AT THESE ZEROS	
N	I	K	ABSOLUTE VALUE	ARGUMENT	ABSOLUTE VALUE	ARGUMENT
4	1	6	3.67023036	0.52505466	0.28598958	-2.88124561
		7	3.89016427	0.52482114	0.26985318	0.26057913
		8	4.08766320	0.52465223	0.25683393	-2.88084537
		9	4.26770478	0.52452436	0.24601085	0.26087472
		10	4.43370345	0.52442418	0.23680815	-2.88061802
4	2	1	1.25505627	1.76052008	0.53711614	0.67640886
		2	2.26657314	1.59473053	0.32536980	-2.37920444
		3	2.76452759	1.58385600	0.26800164	0.77250581
		4	3.12701254	1.57979890	0.23720146	-2.36514064
		5	3.42034992	1.57766926	0.21695303	0.77855062
		6	3.67038911	1.57635565	0.20221649	-2.36174030
		7	3.89028283	1.57546405	0.19080923	0.78073847
		8	4.08775570	1.57481910	0.18160492	-2.36021212
		9	4.26777933	1.57433084	0.17395291	0.78186714
		10	4.43376503	1.57394833	0.16744633	-2.35934401
5	1	1	1.17023190	0.49197486	0.73788630	0.19968410
		2	2.10498408	0.45271092	0.41924569	-2.92188880
		3	2.49756627	0.45092690	0.34031050	0.22177436
		4	2.77623770	0.45026482	0.29852995	-2.91901381
		5	2.99821577	0.44991777	0.27129226	0.22300622
		6	3.18522966	0.44970382	0.25158868	-2.91832146
		7	3.34817063	0.44955864	0.23640913	0.22345153
		8	3.49336577	0.44945365	0.22420928	-2.91801048
		9	3.62485226	0.44937417	0.21410106	0.22368118
		10	3.74538031	0.4493119	0.20553121	-2.91783386
5	2	1	1.21849446	1.47849557	0.57588393	0.58912099
		2	2.10564782	1.36019593	0.33092584	-2.48495960
		3	2.49779516	1.35390716	0.26868422	0.66393331
		4	2.77635789	1.35157105	0.23571083	-2.47482066
		5	2.99829124	1.35034619	0.21420942	0.66828033
		6	3.18528207	1.34959102	0.19865382	-2.47237706
		7	3.34820944	1.34907859	0.18666920	0.66985213
		8	3.49339585	1.34870796	0.17703682	-2.47127935
		9	3.62487634	1.34842741	0.16905573	0.67066279
		10	3.74540009	1.34820763	0.16228919	-2.47065588