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Polynomial Expansions*

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Abstract. The expansion of arbitrary power series in various classes of polynomial sets is considered. Some applications are also given.

Notations. We will use the following contracted notation for the generalized hypergeometric function

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right) = {}_pF_q\left(\begin{matrix} a_P \\ b_Q \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_P)_k}{(b_Q)_k} \cdot \frac{z^k}{k!}, $$

where

$$ (a_P)_k \equiv \prod_{j=1}^{p} (a_j)_k, \quad (b_Q)_k \equiv \prod_{j=1}^{q} (b_j)_k \quad \text{and} \quad (\sigma)_k = \frac{\Gamma(\sigma + k)}{\Gamma(\sigma)}. $$

1. Introduction. Recently there has been some interest in establishing expansion formulae of the type

$$ F(zw) = \sum_{n=0}^{\infty} z^n R_n(z) P_n(w), \tag{1.1} $$

where $F(z)$, $R_n(z)$ are power series and the $P_n(w)$ are polynomials of degree at most n. For example, Fields and Wimp [5] proved

$$ \begin{aligned} {}_{p+r}F_{q+s}\left(\begin{matrix} a_P, c_R \\ b_Q, d_S \end{matrix}\middle| zw\right) = \sum_{n=0}^{\infty} & \frac{(a_P)_n(\alpha)_n(\beta)_n}{(b_Q)_n(\gamma+n)_n} \frac{(-z)^n}{n!} \\ & \times {}_{p+2}F_{q+1}\left(\begin{matrix} n+\alpha, n+\beta, n+a_P \\ 1+2n+\gamma, \ n+b_Q \end{matrix}\middle| z\right) \\ & \times {}_{r+2}F_{s+2}\left(\begin{matrix} -n, n+\gamma, c_R \\ \alpha, \beta, d_S \end{matrix}\middle| w\right), \end{aligned} \tag{1.2} $$

while Verma [13] generalized (1.2) to

$$ \begin{aligned} \sum_{m=0}^{\infty} a_m b_m \frac{(zw)^m}{m!} = \sum_{n=0}^{\infty} & \frac{(-z)^n}{n!(\gamma+n)_n} \sum_{r=0}^{\infty} \frac{(\alpha)_{n+r}(\beta)_{n+r}}{r!(\gamma+2n+1)_r} b_{n+r} z^r \\ & \times \sum_{s=0}^{n} \frac{(-n)_s}{s!} \frac{(n+\gamma)_s}{(\alpha)_s(\beta)_s} a_s w^s, \end{aligned} \tag{1.3} $$

and obtained an analogous expansion in two variables. He also gave a q-analogue of

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(1.3) which reduces to (1.3) as $q \longrightarrow 1$. In [12] Verma generalized a result of Niblett [9] first to

$$
\begin{aligned}
{}_{p+s}F_q\left(\begin{matrix} a_P, b_S \\ c_Q \end{matrix}\middle| zw\right) = h \sum_{n=0}^{\infty} \frac{(h-n\alpha+1)_{n-1}}{n!(c_Q)_n} (b_S)_n (e_U)_n (-z)^n \\
\times {}_{s+u+1}F_q\left(\begin{matrix} n+b_S, n+e_U, h+n(1-\alpha) \\ n+c_Q \end{matrix}\middle| z\right) \\
\times {}_{p+2}F_{u+2}\left(\begin{matrix} -n, a_P, 1+h(1-\alpha)^{-1} \\ h-n\alpha+1, e_U, h(1-\alpha)^{-1} \end{matrix}\middle| w\right),
\end{aligned}
\tag{1.4}
$$

and then to

$$
\begin{aligned}
&\sum_{m=0}^{\infty} c_m d_m \frac{(zw)^m}{m!} \\
&= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} [h+k(1-\alpha)]\, c_k w^k \\
&\quad \times \sum_{s=0}^{\infty} (e_U+k)_{s+n-k}(h+k+1-n\alpha)_{s+n-k} d_{s+n} \frac{z^s}{s!}
\end{aligned}
\tag{1.5}
$$

which, he observes [14], contains most of the results of Brown [2], [3], Carlitz [4], Srivastava [11] and Zeitlin [17]. Niblett's result is (1.4) with $w = 1$, $q = r + t$ and $u = r$. Other results of this type are collected in [8].

We note that all the expansions (1.2)–(1.5) are of the type (1.1). The purpose of this work is to show how such expansions can be built up from relatively simple identites. In particular we will show (Section 2) that these "identities" are easily characterized when the $P_n(z)$ are defined by a generating function of Boas and Buck type [1], [10]. It will become apparent that all the formulae (1.2)–(1.5) correspond to special choices for the generating function.

In [7], Ismail showed how to obtain generating functions of Boas and Buck type for any given orthogonal set of polynomials $P_n(w)$. Thus the results of Section 2 are valid for all orthogonal polynomial sets.

Sections 3 and 4 contain applications of Section 2.

2. Fundamental Relationships. First, assume that the $P_n(w)$ are defined by the Boas and Buck generating function

$$
A(t)\Phi(wH(t)) = \sum_{n=0}^{\infty} P_n(w)t^n, \qquad H(0) = 0, \qquad H'(0)A(0) \neq 0,
\tag{2.1}
$$

where the $A(t)$, $H(t)$ and $\Phi(t)$ are power series in t satisfying the indicated requirements in (2.1). Under these requirements we can make the change of variable $u = H(t)$ and rewrite (2.1) in the form

$$
\Phi(wu) = \sum_{n=0}^{\infty} P_n(w)\{t(u)\}^n / A(t(u)).
$$

Setting

$$\{t(u)\}^n/A(t(u)) = \sum_{j=0}^{\infty} \lambda_{n,j} u^{n+j}, \qquad n = 0, 1, \ldots,$$

and

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we note that $\lambda_{n,0} \neq 0$, and that equating coefficients of u^m, we get

$$a_m w^m = \sum_{n=0}^{m} \lambda_{n,m-n} P_n(w), \qquad m = 0, 1, \ldots. \tag{2.2}$$

Multiplying (2.2) by $b_m z^m$ and summing over m, we formally obtain

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{m=0}^{\infty} b_m z^m \sum_{n=0}^{m} \lambda_{n,m-n} P_n(w) = \sum_{n=0}^{\infty} z^n R_n(z) P_n(w), \tag{2.3}$$

$$R_n(z) = \sum_{m=0}^{\infty} b_{n+m} \lambda_{n,m} z^m. \tag{2.4}$$

Inversely, if we define $\mu_{n,j}$ by

$$A(t)\{H(t)\}^n = \sum_{j=0}^{\infty} \mu_{n,j} t^{n+j}$$

then

$$P_n(w) = \sum_{j=0}^{n} \mu_{j,n-j} a_j w^j. \tag{2.5}$$

Substitution of (2.5) into (2.2), and vice versa, for arbitrary a_m, leads to the equivalent orthogonality relationships

$$\sum_{j=k}^{m} \lambda_{j,m-j} \mu_{k,j-k} = \delta_{m,k}, \qquad m \geqslant k \geqslant 0, \tag{2.6}$$

$$\sum_{j=k}^{m} \mu_{j,m-j} \lambda_{k,j-k} = \delta_{m,k}, \qquad m \geqslant k \geqslant 0. \tag{2.7}$$

In particular,

$$\lambda_{m,0} \mu_{m,0} = 1, \qquad m = 0, 1, \cdots.$$

To see that (2.6) and (2.7) are equivalent, let

$$U = \begin{pmatrix} \mu_{00}, \mu_{01}, \mu_{02}, \ldots \\ 0, \mu_{10}, \mu_{11}, \ldots \\ 0, \quad 0, \mu_{20}, \ldots \\ \cdots\cdots\cdots \end{pmatrix}, \qquad L = \begin{pmatrix} \lambda_{00}, \lambda_{01}, \lambda_{02}, \ldots \\ 0, \lambda_{10}, \lambda_{11}, \ldots \\ 0, \quad 0, \lambda_{20}, \ldots \\ \cdots\cdots\cdots \end{pmatrix}.$$

Then (2.6) and (2.7) can be interpreted as $UL = LU = I$ where I is the infinite identity matrix.

Formal substitution of (2.4) in the following, yields

$$(2.8)\qquad \sum_{n=0}^{\infty} \mu_{m,n} z^{n+m} R_{n+m}(z) = \sum_{n=m}^{\infty} b_n z^n \sum_{j=m}^{n} \mu_{m,j-m} \lambda_{j,n-j} = b_m z^m .$$

For completeness, we note that when (2.8) is multiplied by $a_m w^m$ and summed over m, we again formally obtain (2.3) together with (2.5).

From the above discussion, it is clear that either the "identity" (2.2), when the polynomials $P_n(x)$ are specified, or the "identity" (2.8), when the functions $R_n(x)$ are specified, is sufficient to formally obtain the expansion (2.3). Moreover, once the $\lambda_{n,j}$, $\mu_{n,j}$ have been introduced in the identities (2.2) and (2.8), the subsequent development, including (2.3), is independent of any generating function origin.

From (2.6) and (2.7) it is clear that the $\lambda_{n,j}$ and $\mu_{n,j}$ are to some extent interchangeable, i.e. if we set

$$Q_n(w) = \sum_{j=0}^{n} \lambda_{j,n-j} a_j w^j, \qquad n = 0, 1, \cdots ,$$

then we again have

$$a_m w^m = \sum_{n=0}^{m} \mu_{n,\, m-n} Q_n(w), \qquad m = 0, 1, \cdots ,$$

and formal substitution yields

$$(2.9)\qquad \sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} z^n S_n(z) Q_n(w), \qquad S_n(z) = \sum_{m=0}^{\infty} b_{n+m} \mu_{n,m} z^m, \qquad n = 0, 1, \cdots .$$

We will refer to (2.9) as the dual expansion, and the $Q_n(w)$ as the dual polynomials.

More basically still, we note that if $\lambda_{n,j}$ is any double sequence such that $\lambda_{n,0} \neq 0$ for all n, and that if the $\mu_{n,j}$ are chosen to satisfy (2.6) or (2.7), as they always can be, then (2.2) and (2.8) can be derived formally from (2.6) or (2.7), i.e. the computation in (2.8) formally derives (2.8) from (2.6), while substitution of (2.5) into (2.2) derives (2.2) from (2.6).

It is worth mentioning that only those functions $F(zw) = \Sigma_{n=0}^{\infty} c_n (zw)^n$ can be expanded in the form (2.3) which satisfy the requirement that $c_n = 0$ if and only if $a_n b_n = 0$.

Multidimensional analogues can be similarly obtained. Let $P_n(w)$ and $W_n(v)$ satisfy

$$a_n w^n = \sum_{k=0}^{n} \lambda_{k,n-k} P_k(w), \qquad b_n v^n = \sum_{l=0}^{n} \gamma_{l,n-l} W_l(v).$$

Then formally, we have

$$\sum_{m,n=0}^{\infty} a_m b_n c_{m,n} (zw)^m (vy)^n$$

$$= \sum_{l,k=0}^{\infty} W_l(v) P_k(w) z^k y^l \sum_{m,n=0}^{\infty} \gamma_{l,n} \lambda_{k,m} c_{m+k,\, l+n} z^m y^n$$

which is a two dimensional analogue of (2.3). The extension to higher dimensions is immediate. Similarly q-analogues in one and several variables may be derived.

3. Remarks and Examples. An interesting feature of (2.2) is that the $P_n(w)$ need not form a basic set of polynomials [1] as in Fields and Wimp [6]. An obvious feature of (1.1) is that if $d_0, d_1, \dots, d_n, \dots$ is a sequence of nonzero numbers then replacing a_n, b_n by a_n/d_n, $b_n d_n$, respectively, introduces new factors in $P_n(w)$ and $R_n(z)$ but does not change the left-hand side of (1.1). This is the origin of the free parameters α and β in Fields and Wimp's expansion (1.2) and in Verma's (1.3). With this in mind, we note that the parameters e_U of (1.5) are redundant since $(e_U + k)_{n+s-k}$ is nothing but $(e_U)_{n+s}/(e_U)_k$.

We now proceed with some examples.

Example 1. Take $H(t) = -4t(1-t)^{-2}$, $A(t) = (1-t)^{-c}$ and $\Phi(z) = \Sigma_{n=0}^{\infty} a_n z^n$. Then $u(t) = -4t(1-t)^{-2}$ implies $t = -u(1+\sqrt{1-u})^{-2}$ and simple computations lead to (see [10, pp. 137–140])

$$P_n(w) = \frac{(c)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (c+n)_k}{(c)_{2k}} a_k (4w)^k,$$

and

$$a_n w^n = \frac{(c)_{2n}}{4^n n!} \sum_{k=0}^{n} \frac{(-n)_k (c+2k)}{(c)_{n+k+1}} P_k(w).$$

Therefore,

$$\sum_{n=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} (c+2n) \frac{(c)_n}{n!} (-z)^n \sum_{j=0}^{\infty} \frac{(c/2)_{n+j}((c+1)/2)_{n+j}}{j!(c)_{2n+j+1}} b_{n+j} z^j \tag{3.1}$$
$$\times \sum_{k=0}^{n} \frac{(-n)_k (c+n)_k}{(c)_{2k}} a_k (4w)^k.$$

Replacing a_n, b_n by

$$\frac{(c/2)_n((c+1)/2)_n}{n!} a_n, \qquad \frac{1}{(c/2)_n((c+1)/2)_n} b_n$$

respectively, we get essentially Verma's formula (1.3).

The dual computations then lead to

$$Q_n(w) = \frac{4^{-n}(c)_{2n}}{n!(c)_{n+1}} \sum_{k=0}^{n} \frac{(-n)_k (c+2k)}{(c+n+1)_k} a_k w^k$$

and

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} \frac{(c)_{2n}}{n!(c)_{n+1}} (-z)^n \sum_{j=0}^{\infty} \frac{(c+2n)_j}{j!} b_{n+j} z^j$$
$$\times \sum_{k=0}^{n} \frac{(-n)_k (c+2k)}{(c+n+1)_k} a_k w^k.$$

Example 2. Let $H(t) = -t(1-t)^{-1}$, $A(t) = (1-t)^{-c}$ and $\Phi(z) = \Sigma_{n=0}^{\infty} a_n z^n$.

Then formulae (2.5) and (2.2) reduce to

$$P_n(w) = \frac{(c)_n}{(1)_n} \sum_{k=0}^{n} \frac{(-n)_k a_k}{(c)_k} w^k,$$

and

$$a_n w^n = \frac{(c)_n}{(1)_n} \sum_{k=0}^{n} \frac{(-n)_k}{(c)_k} P_k(w),$$

respectively. Proceeding as in Section 2, we arrive at

$$(3.2)\quad \sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} (-z)^n \sum_{j=0}^{\infty} \frac{(n+c)_j}{j!} b_{n+j} z^j \sum_{k=0}^{n} \frac{(-n)_k}{(c)_k} a_k w^k,$$

which is a generalization of Fields and Wimp's expansion

$$(3.3)\quad \begin{aligned} &{}_{p+r}F_{q+s}\left(\begin{matrix} a_P, c_R \\ b_Q, d_S \end{matrix} \middle| zw\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_P)_n (\alpha)_n (-z)^n}{(b_Q)_n n!} {}_{p+1}F_q\left(\begin{matrix} n+\alpha, n+a_P \\ n+b_Q \end{matrix} \middle| z\right) {}_{r+1}F_{s+1}\left(\begin{matrix} -n, c_R \\ \alpha, d_S \end{matrix} \middle| w\right). \end{aligned}$$

In [5], Fields and Wimp derived (3.3) from (1.2) by confluence. Similarly, we could derive (3.2) from (3.1).

The expansion (3.2) is selfdual.

Example 3. Brown [3] proved that the polynomials

$$(3.4)\quad P_n(w) = \sum_{k=0}^{n} \binom{a+bn}{n-k} a_k w^k,$$

are generated by

$$(3.5)\quad \sum_{n=0}^{\infty} \frac{a}{a+bn} P_n(w) \left\{\frac{u}{(1+u)^b}\right\}^n = (1+u)^a \sum_{n=0}^{\infty} \frac{a}{a+bn} a_n (wu)^n.$$

The generating function (3.5) is clearly of Boas and Buck type. The corresponding $H(t)$ and $A(t)$ are defined implicitly by $H(t) = t[1 + H(t)]^b$ and $A(t) = (1 + H(t))^a$. The relationship (3.5) implies

$$(3.6)\quad a_n w^n = \sum_{j=0}^{n} \binom{-a-bj}{n-j} \frac{(a+bn)}{(a+bj)} P_j(w).$$

Thus we have

$$(3.7)\quad \begin{aligned} \sum_{m=0}^{\infty} a_m b_m (zw)^m &= \sum_{n=0}^{\infty} \frac{z^n}{a+bn} \sum_{j=0}^{\infty} (a+bn+bj) b_{n+j} \binom{-a-bn}{j} z^j \\ &\quad \times \sum_{k=0}^{n} \binom{a+bn}{n-k} a_k w^k, \end{aligned}$$

which is essentially Verma's (1.5). The dual expansion of (3.7),

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} (a+bn)z^n \sum_{j=0}^{\infty} \binom{a+bn+bj}{j} b_{n+j} \frac{z^j}{a+bn+bj} \times \sum_{k=0}^{n} \binom{-a-bk}{n-k} a_k w^k,$$

follows easily from (3.4) and (3.6).

For the sake of completeness we include a simple proof of (3.5). Clearly (3.5) is equivalent to (3.6), which in turn is equivalent to the orthogonality relation

$$(3.8) \qquad (a+bl)\delta_{n,0} = \sum_{k=0}^{n} \binom{a+bn+bl}{n-k}\binom{-a-bl}{k}(a+bk+bl).$$

The relationship (3.8) is obvious for $n = 0$. For $n > 0$, its right-hand side is equal to

$$(a+bl)\binom{a+bn+bl}{n} {}_2F_1\left(\begin{matrix} -n, a+bl \\ a+bn+bl+1-n \end{matrix}\middle| 1\right) - b(a+bl)\binom{a+bn+bl}{n-1} {}_2F_1\left(\begin{matrix} -n+1, a+bl+1 \\ a+bn+bl+2-n \end{matrix}\middle| 1\right),$$

and hence is zero by Gauss's theorem.

Example 4. Consider the case $A(t) = (1+t^2)^{-\nu}$, $H(t) = 2t/(1+t^2)$, and $\Phi(z) = \Sigma_0^\infty a_n z^n$, which includes the Gegenbauer (ultraspherical) polynomials [10] $c_n^{(\nu)}(w)$ as the special case $\Phi(z) = (1-z)^{-\nu}$. Let

$$(3.9) \qquad \sum_0^\infty P_n(w)t^n = (1+t^2)^{-\nu}\Phi\left(\frac{2tw}{1+t^2}\right).$$

The explicit representation

$$(3.10) \qquad P_n(w) = \sum_{k=0}^{n} \binom{2k-n-\nu}{k} a_{n-2k}(2w)^{n-2k}, \qquad a_l = 0 \text{ if } l < 0,$$

follows easily from (3.9). In (3.9) let $u = 2t/(1+t^2)$, or $t = u(1+\sqrt{1-u^2})^{-1}$; and using

$${}_2F_1(\gamma, \gamma - \tfrac{1}{2}; 2\gamma; z) = \left\{\frac{2}{1+\sqrt{1-z}}\right\}^{2\gamma-1}$$

[10, p. 70], we get

$$(3.11) \qquad 2^m a_m w^m = (\nu)_m \sum_{j=0}^{m} \frac{(\nu+m-2j)}{j!(\nu)_{m+1-j}} P_{m-2j}(w).$$

Thus we have the new expansion

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} \frac{(\nu+n)}{2^n} z^n \sum_{j=0}^{\infty} \frac{(\nu+n+j)_j (z/2)^{2j}}{j!(\nu+n+j)} b_{n+2j} \times \sum_{k=0}^{m} \binom{2k-n-\nu}{k} a_{n-2k}(2w)^{n-2k}$$

and its dual

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} (\nu)_n z^n \sum_{k=0}^{\infty} (-1)^k \frac{(n+\nu)_k}{k!} b_{n+2k} z^{2k} \times \sum_{j=0}^{[n/2]} \frac{(\nu+n-2j)}{j!(\nu)_{n+1-j}} a_{n-2j} w^{n-2j}.$$

4. **Applications.** The kernel function $1/(z-w)$ is one of the most important functions in complex function theory. If in (1.1) we take $a_n b_n = 1$, we get

$$\frac{1}{1-zw} = \sum_{n=0}^{\infty} z^n R_n(z) P_n(w),$$

or

$$\frac{1}{w-z} = w^{-1} \sum_{n=0}^{\infty} z^n R_n(z) P_n\left(\frac{1}{w}\right) \quad \text{and} \quad \frac{1}{z-w} = \sum_{n=0}^{\infty} z^{-n-1} R_n\left(\frac{1}{z}\right) P_n(w). \tag{4.1}$$

Using Cauchy's Theorem to represent an arbitrary analytic function $f(z)$ as a contour integral, substituting into the integral (4.1) and formally interchanging the order of summation and integration, one obtains the formal expansions

$$f(z) = \sum_{n=0}^{\infty} z^n R_n(z) \cdot \frac{1}{2\pi i} \int_c f(w) w^{-1} P_n(w^{-1})\, dw$$

and

$$f(w) = \sum_{n=0}^{\infty} P_n(w) \cdot \frac{1}{2\pi i} \int_c f(z) z^{-n-1} R_n(z^{-1})\, dz,$$

for some appropriate contour c.

We will not consider the convergence of such expansion problems for analytic functions here.

Formulas of the type (1.1) are particularly useful in expansions of convolution transforms. Let

$$[Tf; x] = \int_{-\infty}^{\infty} K(xt) f(t)\, dt, \qquad K(zw) = \sum_{n=0}^{\infty} z^n R_n(z) P_n(w) \tag{4.2}$$

be such a transform. We formally have the polynomial expansion

$$[Tf; w] = \sum_{n=0}^{\infty} P_n(w) \int_{-\infty}^{\infty} f(t) t^n R_n(t)\, dt, \tag{4.3}$$

as well as the expansion

$$[Tf; z] = \sum_{0}^{\infty} z^n R_n(z) \int_{-\infty}^{\infty} f(t) P_n(t)\, dt. \tag{4.4}$$

In [15] and [16] Wimp established the expansion (4.3) for the Laplace transform and some other special transforms. He also discussed the merits of such expansions in numerical computations. Expansions of the type (4.4) are also important when the $R_n(z)$ can be efficiently computer generated as in the case when $R_n(z) = z^{-(n+\nu)/2} I_{n+\nu}(\sqrt{z})$, the modified Bessel function of the second kind.

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