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# An Equation of Mordell

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**Abstract.** All integer solutions of the Diophantine equation $6y^2 = (x + 1)(x^2 - x + 6)$ are found.

1. Mordell [1] asks if all the integer solutions of the Diophantine equation $6y^2 = (x + 1)(x^2 - x + 6)$ are given by $x = -1, 0, 2, 7, 15$ and $74$. It is shown that there are precisely seven integer solutions, the seventh with $x = 767$.

Consideration of factorization gives, for some integers $a$, $b$,

$$\begin{cases} x^2 - x + 6 = 6b^2 \\ x + 1 = a^2 \end{cases} \text{or} \quad \begin{cases} x^2 - x + 6 = 3b^2 \\ x + 1 = 2a^2 \end{cases} \text{or}$$

$$\begin{cases} x^2 - x + 6 = 2b^2 \\ x + 1 = 3a^2 \end{cases} \text{or} \quad \begin{cases} x^2 - x + 6 = b^2 \\ x + 1 = 6a^2 \end{cases}$$

and the latter case is impossible modulo 3. We thus obtain, on eliminating $x$, the three quartic equations,

(i) $a^4 - 3a^2 + 8 = 6b^2$,
(ii) $4a^4 - 6a^2 + 8 = 3b^2$,
(iii) $9a^4 - 9a^2 + 8 = 2b^2$.

The standard technique in dealing with equations of this type is to factorize in the appropriate quadratic extension of the integers, which here is $\mathbf{Z}[(1 + \sqrt{-23})/2]$, to obtain a finite set of equations of the form,

$$a^2 = f(v, w), \qquad 1 = g(v, w),$$

where $f$, $g$ are homogeneous quadratic forms.

We need to know some details of the quadratic field $\mathbf{Q}(\sqrt{-23})$. The class-number of the ring of integers is 3; and we denote the ideal factorizations of 2 and 3 by $(2) = \mathfrak{p}_2\bar{\mathfrak{p}}_2$, $(3) = \mathfrak{p}_3\bar{\mathfrak{p}}_3$ where a bar denotes conjugacy, and $\mathfrak{p}_2\mathfrak{p}_3 = ((1 + \sqrt{-23})/2)$.

Thus in Eq. (i), $(2a^2 - 3)^2 + 23 = 24b^2$ implies the ideal equation

$$\left(\frac{2a^2 - 3 \pm \sqrt{-23}}{2}\right) = \mathfrak{q}\mathfrak{b}^2 \quad \text{where } \mathfrak{q}\bar{\mathfrak{q}} = (6) \text{ and } \mathfrak{b} \text{ is some integral ideal.}$$

There are essentially two possibilities, $\mathfrak{q} = \mathfrak{p}_2\mathfrak{p}_3$ and $\mathfrak{q} = \bar{\mathfrak{p}}_2\mathfrak{p}_3$. In the former instance, $\mathfrak{b}$ is principal, and in the latter, $\mathfrak{b}\bar{\mathfrak{p}}_2$ is principal.

Since $\mathfrak{p}_3\bar{\mathfrak{p}}_2^{-1} = ((1 + \sqrt{-23})/4)$ we have, respectively,

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$$\pm\left(\frac{2a^2-3\pm\sqrt{-23}}{2}\right)=\left(\frac{1+\sqrt{-23}}{2}\right)\left(\frac{u+v\sqrt{-23}}{2}\right)^2$$

and

$$\pm\left(\frac{2a^2-3\pm\sqrt{-23}}{2}\right)=\left(\frac{1+\sqrt{-23}}{4}\right)\left(\frac{u+v\sqrt{-23}}{2}\right)^2$$

for some integers $u$, $v$ satisfying $u \equiv v \mod 2$. Thus we have, respectively,

$$\begin{cases} -(2a^2-3) = \dfrac{u^2-46uv-23v^2}{4} \\ \quad 1 = \dfrac{u^2+2uv-23v^2}{4} \end{cases}$$

and

$$\begin{cases} (2a^2-3) = \dfrac{u^2-46uv-23v^2}{8} \\ \quad -1 = \dfrac{u^2+2uv-23v^2}{8} \end{cases}$$

where the signs in each equation have been determined by a congruence modulo 3.

In the former case, putting $u + v = 2w$, we obtain

$$\text{I:}\ \begin{cases} a^2 = w^2 + 12wv - 12v^2 \\ 1 = w^2 - 6v^2. \end{cases}$$

In the latter case, $u^2 + 2uv + v^2 \equiv 0 \mod 8$, so $u + v = 4w$ say; then

$$\text{II:}\ \begin{cases} a^2 = 6v^2 - 12vw - 2w^2 \\ 1 = 3v^2 - 2w^2. \end{cases}$$

In similar manner (ii) gives rise to

$$\text{III:}\ \begin{cases} a^2 = -2w^2 + 12wv - 6v^2 \\ 1 = 4w^2 - 3v^2 \end{cases}$$

and (iii) to the three pairs

$$\text{IV:}\ \begin{cases} a^2 = v(9v+16w) \\ 1 = 9v^2 - 8w^2 \end{cases} \qquad \text{V:}\ \begin{cases} a^2 = v(v+8w) \\ 1 = v^2 - 18w^2 \end{cases}$$

and

$$\text{VI:}\ \begin{cases} a^2 = 32w(v-9w) \\ 1 = v^2 - 288w^2. \end{cases}$$

Of these six pairs of equations, V and VI may be treated by simple descent arguments. For instance, in VI, we have that $w = m^2$ or $2m^2$ after change of sign if necessary: so it suffices to determine all integer solutions of the equations $1 = v^2 - 18m^4$ and $1 = v^2 - 72m^4$, respectively. This is readily achieved by means of a classical descent argument; but we can quote Ljunggren [2] to say that the only integer solutions of the former are $(\pm r, \pm m) = (1, 0)$ and $(17, 1)$, and of the latter $(\pm r, \pm m) = (1, 0)$. These give the solutions $a = 0$ and $a = 16$ of Eq. (iii) whence solutions $x = -1, 767$ of the original equation.

Each of the four remaining pairs of equations represents the intersection of two quadrics in three-dimensional space; the method of solution, as exploited by Cassels [3], is to consider the singular elements in the pencil of the quadrics. Such singular quadrics are given by $f - \lambda g$, where $\det(f - \lambda g) = 0$: that is, a linear combination of $f$ and $g$ which is a perfect square. In general, of course, $\lambda$ is a quadratic irrational. We can thus rewrite each pair of equations in the form $a^2 - \mu L(v, w)^2 = \lambda$ for some $\mu \in \mathbf{Q}(\lambda)$, where $L(v, w)$ is a homogeneous linear form with coefficients in $\mathbf{Q}(\lambda)$. We now work over $\mathbf{Q}(\delta)$ where $\delta^2 = \mu$ and equate $(a + L\delta)$ and $(a - L\delta)$ as ideals, to two ideal factors of $\lambda$ in $\mathbf{Q}(\delta)$, noting that the two factors must be conjugate over $\mathbf{Q}(\mu)$. All the ideals are principal, so using the appropriate arithmetical details of the field $\mathbf{Q}(\delta)$, we can equate coefficients of elements of an integer base; in particular, it is clear that the coefficient of $\delta^2$ in $a + L\delta$ is zero, and the resulting equation is completely solved by congruence considerations.

As an illustration, consider Eq. II. The singular quadrics in this pencil are obtained by taking a linear combination which is a perfect square: so let $3(2 + \lambda)v^2 - 12wv - 2(1 + \lambda)w^2$ be a perfect square. Then $36 = -6(1 + \lambda)(2 + \lambda)$ or $\lambda^2 + 3\lambda + 8 = 0$. Taking $\lambda = (-3 - \sqrt{-23})/2$ we obtain

$$a^2 - (1 + \sqrt{-23})\left[w - \frac{1 - \sqrt{-23}}{4}v\right]^2 = \frac{3 + \sqrt{-23}}{2},$$

and accordingly work in $\mathbf{Q}(\delta)$ where $\delta^2 = 1 + \sqrt{-23}$. We need some arithmetical details of this field; certainly $\tau = (\delta^3 - 2\delta^2 + 2\delta + 4)/8$ is an algebraic integer, since $\tau^2 - \tau((1 - \sqrt{-23})/2) + 1 = 0$. The discriminant of $R = \mathbf{Z}[1, \delta, \delta^2/2, \tau]$ is $2^3 \cdot 3 \cdot 23^2$, whence $R$ is indeed the ring of integers of the field (for 23 certainly ramifies, so $23^2$ divides the discriminant, and Stickelberger's criterion says that the discriminant is congruent to 0 or 1 modulo 4). It is also readily calculated by standard techniques that $\tau$ is a fundamental unit for the field, and that we have the factorization, $(2) = \mathfrak{q}_2(\mathfrak{q}_2')^2$, where $\mathfrak{q}_2 = \mathfrak{p}_2$, $(\mathfrak{q}_2')^2 = \bar{\mathfrak{p}}_2$, with $\mathfrak{p}_2 = (2, (1 + \sqrt{-23})/2)$, $\bar{\mathfrak{p}}_2 = (2, (1 - \sqrt{-23})/2)$.

The equation now becomes in terms of ideals,

$$\left(a + \delta\left(w - \frac{v}{2}\right) + \frac{\delta^3}{4}v\right)\left(a - \delta\left(w - \frac{v}{2}\right) - \frac{\delta^3}{4}v\right) = (\mathfrak{q}_2')^6;$$

and since the two ideals on the left are conjugate over $\mathbf{Z}[(1 + \sqrt{-23})/2]$ we must have

$$\left(a + \delta\left(w - \frac{v}{2}\right) + \frac{\delta^3}{4}v\right) = (\mathfrak{q}_2')^3 = \left(\frac{\delta^3 - 6\delta + 16}{4}\right).$$

Because there are no nontrivial roots of unity in $\mathbf{Q}(\delta)$ we now obtain $a + \delta(w - v/2) + (\delta^3/4)v = \pm((\delta^3 - 6\delta + 16)/4)\tau^n$ for some integer $n$. This exponential equation is solved by first comparing coefficients of $\delta^2$, using the fact that $\tau^5 \equiv -1 \bmod 7$; a congruence modulo a suitable power of 7 then shows that the only solutions are given by $n = 0$ or $-3$. These give $a = 4$ as solution of (i), and $x = 15$ as a solution of the original equation.

The complete details of the proof are to appear in my Ph.D. Thesis. I gratefully thank Professors Swinnerton-Dyer and Cassels for their advice and encouragement.

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