51 [9.00].—ROBERT BAILLIE, *Data on Artin's Conjecture*, Computer-Based Education Research Laboratory, University of Illinois, Urbana, Illinois, 1975, xlvi + about 280 pp. of computer output followed by 108 pp. of hard copy summaries deposited in the UMT file.

This is by far the most extensive data known to me that supports Artin's primitive root conjecture as it was modified by the Lehmers [1] and by Heilbronn, cf. [2, §23.2]. Let $\nu_a(x)$ be the number of primes $\leqslant x$ having a as a primitive root and let $\pi(x)$ be the number of primes $\leqslant x$. For $a = \pm 2, \pm 3, -4, \pm 5, \pm 6, \pm 7, \pm 8, -9, \pm 10, \pm 11, \pm 12, \pm 13$ the UMT deposited here strongly supports the conjecture

$$\nu_a(x) \sim f_a A\pi(x) \tag{1}$$

where A is given in (2) of the previous review, and $f_a = 1$ for these a except in these cases: $f_{-3} = f_{-12} = 6/5, f_5 = 20/19, f_{-7} = 42/41, f_{-8} = f_8 = 3/5, f_{-11} = 110/109$ and $f_{13} = 156/155$. (The original Artin conjecture had an error in that it set all of these $f_a = 1$ except for $a = \pm 8$; cf. [6] of the previous review.)

The main table is in two parts: the first is for the ten positive values of a, and then, at this reviewer's suggestion, the twelve negative a were also computed. For $x = 25 \cdot 10^3(25 \cdot 10^3)33 \cdot 10^6$ there are listed here x, $\pi(x)$, the counts $\nu_a(x)$, the approximations $\langle f_a A\pi(x)\rangle$ rounded to the nearest integer, the empirical ratio $\nu_a(x)/\pi(x)$ to 8D and the ratio-differences $[\langle f_a A\pi(x)\rangle - \nu_a(x)]/\pi(x)$ also to 8D. But note that the rounding of $f_a A\pi(x)$ in the last quantity affects it considerably; the use of unrounded approximations would change these 8D differences substantially. These extensive tables required seventy-one hours of idle CPU time on a CDC Cyber-73.

The foregoing data was then summarized in four different ways on a PLATO IV terminal that produced the hard copies included here. Each of the thirty-three pages of Part I of these summaries lists the data for a single value of $x = n \cdot 10^6$ $(n = 1$ to $33)$ versus all twenty-two values of a. It lists $\nu_a(x)$, $\langle f_a A\pi(x)\rangle$ and their difference

$$d_a(x) = \nu_a(x) - \langle f_a A\pi(x)\rangle. \tag{2}$$

Each of the twenty-two pages of Part II lists the data for a single a versus the thirty-three values of x. Part III is a plot of the noncumulative differences in each interval of 10^6 while Part IV plots the cumulative differences $d_a(x)$ versus x with a fixed. Part IV also lists the maximal $d_a(x)$ that occurs up to $33 \cdot 10^6$ and the distribution of the thirty-three values of $d_a(x)$ according to the sign of $d_a(x)$.

For most a, $d_a(x)$ is usually > 0. See the previous review for a discussion of the case $a = 10$. In Part IV, $d_{10}(x) > 0$ thirty-three times and < 0 never. But at $x = 150000$ in the original data there is a short interval when $d_{10}(x) < 0$; specifically, $\nu_{10}(150000) = 5167$ and $\langle f_{10} A\pi(x)\rangle = 5179$. For $a = -8, -10$ and -13, negative $d_a(x)$ predominate.

For all twenty-two a and all thirty-three $x = n \cdot 10^6$ I find the empirical law

$$|d_a(x)| < \sqrt{f_a A\pi(x)}, \tag{3}$$

i.e., the difference is less than the square-root of the expected value. On the other hand,

$$|d_a(x)| > \tfrac{1}{2}\sqrt{f_a A\pi(x)}$$

occurs frequently, so if (3) is true for all $x >$ some small x_0, then the right side of (3) would be close to the best possible bound. A weakened version of (3), namely,

$$d_a(x) = O\left(\frac{x}{\log x}\right)^{1/2+\epsilon} \tag{4}$$

would not only imply (1) but also that the error term is smaller than has been previously suggested. Hooley's theory ([5] of the previous review) deduces the much larger error

$$O\left(\frac{x}{\log^2 x} \log \log x\right)$$

from his hypotheses. Perhaps someone should review his calculation to see if a smaller error term is not obtainable.

For earlier, far less extensive tables see [1]–[3] and the references cited there. The program used in these computations is also deposited.

D. S.

1. D. H. LEHMER & EMMA LEHMER, "Heuristics anyone?," *Studies in Mathematical Analysis and Related Topics*, Stanford Univ. Press, Stanford, Calif., 1962, pp. 202–210.

2. A. E. WESTERN & J. C. P. MILLER, *Indices and Primitive Roots*, Cambridge Univ. Press, 1968. Reviewed in *Math. Comp.*, v. 23, 1969, pp. 683–685.

3. J. C. P. MILLER, *Primitive Root Counts*, UMT 54, *Math. Comp.*, v. 26, 1972, p. 1024.