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A Quadratically Convergent Iteration Method for Computing Zeros of Operators Satisfying Autonomous Differential Equations

By L. B. Rall*

Abstract. If the Fréchet derivative P' of the operator P in a Banach space X is Lipschitz continuous, satisfies an autonomous differential equation $P'(x) = f(P(x))$, and $f(0)$ has the bounded inverse Γ, then the iteration process

$$x_{n+1} = x_n - \Gamma P(x_n), \qquad n = 0, 1, 2, \ldots,$$

is shown to be locally quadratically convergent to solutions $x = x^*$ of the equation $P(x) = 0$. If f is Lipschitz continuous and Γ exists, then the global existence of x^* is shown to follow if $P(x)$ is uniformly bounded by a sufficiently small constant. The replacement of the uniform boundedness of P by Lipschitz continuity gives a semilocal theorem for the existence of x^* and the quadratic convergence of the sequence $\{x_n\}$ to x^*.

Successive approximations $x_1, x_2, \ldots$ to a solution $x = x^*$ of the operator equation $P(x) = 0$ in a Banach space X can be obtained under suitable conditions from an iteration process of the form

$$x_{n+1} = x_n - [P'(y_n)]^{-1}P(x_n), \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where the initial approximation x_0 and the sequence $\{y_n\}$ are given, and the existence of the inverses of the (Fréchet) derivatives $\{P'(y_n)\}$ and the convergence of the sequence $\{x_n\}$ to x^* can be guaranteed. Special cases of (1) are *Newton's method* ($y_n = x_n$) and the *modified* Newton's method ($y_n = x_0$); so methods of this type may be characterized as *variants* of Newton's method, or *Newton-like* methods ([2], [3]).

1. **Local Convergence.** It will be assumed that $P(x^*) = 0$ and $\|P'(x) - P'(y)\| \leqslant K\|x - y\|$, at least in a sufficiently large region containing x^*. The inequality [4]

$$\|x_{n+1} - x^*\| \leqslant \tfrac{1}{2}K\|[P'(y_n)]^{-1}\|\{\|x_n - y_n\| + \|y_n - x^*\|\}\|x_n - x^*\| \tag{2}$$

is useful for estimating the rate of convergence of $\{x_n\}$ to x^*. If one takes $y_n = \lambda_n x_n + (1 - \lambda_n)x^*$, $0 \leqslant \lambda_n \leqslant 1$, then $\|x_n - y_n\| + \|y_n - x^*\| = \|x_n - x^*\|$, and one has

$$\|x_{n+1} - x^*\| \leqslant \tfrac{1}{2}K\|[P'(y_n)]^{-1}\| \cdot \|x_n - x^*\|^2, \tag{3}$$

which shows that convergence will be quadratic if the inverses $[P'(y_n)]^{-1}$ are uniformly

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bounded. The method of present interest is obtained by taking $\lambda_n = 0$, so that $y_n = x^*$. If now $\Gamma = [P'(x^*)]^{-1}$ exists and $\|\Gamma\| \leqslant B^*$, then the iteration process

$$x_{n+1} = x_n - \Gamma P(x_n), \qquad n = 0, 1, 2, \ldots, \tag{4}$$

will be quadratically convergent, with

$$\|x_{n+1} - x^*\| \leqslant \tfrac{1}{2} KB^* \|x_n - x^*\|^2. \tag{5}$$

The iteration process (4) has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, as the operator Γ is calculated once and for all. This method can be realized for operators P which satisfy an autonomous differential equation

$$P'(x) = f(P(x)), \tag{6}$$

as $P'(x^*) = f(0)$ can be evaluated without knowing the value of x^*. With the above assumptions one has the following result.

THEOREM 1. *If* $\Gamma = [f(0)]^{-1}$ *exists,* $\|\Gamma\| \leqslant B^*$, *and* x_0 *is such that*

$$\theta = \tfrac{1}{2} KB^* \|x_0 - x^*\| < 1, \tag{7}$$

then the sequence $\{x_n\}$ *defined by* (4) *converges to* x^*, *with*

$$\|x_n - x^*\| \leqslant \theta^{2^n - 1} \|x_0 - x^*\|, \qquad n = 1, 2, \ldots. \tag{8}$$

Proof. Inequality (8) follows from (5) and (7) by mathematical induction.

For example, the iteration process

$$x_{n+1} = x_n - \frac{1}{N}(e^{x_n} - N), \qquad n = 0, 1, 2, \ldots, \tag{9}$$

converges quadratically to the solution $x^* = \ln N$ of $P(x) \equiv e^x - N = 0$ for sufficiently close initial approximations x_0; here (6) is $P'(x) = P(x) + N$.

2. A Global Existence Theorem. It will be assumed that $\Gamma = [f(0)]^{-1}$ exists, $\|\Gamma\| \leqslant B^*$, and conditions for the existence of x^* will be obtained.

THEOREM 2. *If* f *is Lipschitz continuous with constant* α, $\|P(x)\| \leqslant \beta$, *and*

$$\rho = \alpha\beta B^* < 1, \tag{10}$$

then the equation $P(x) = 0$ *has a unique solution* x^* *to which the sequence* $\{x_n\}$ *defined by* (4) *converges, with*

$$\|x^* - x_n\| \leqslant \frac{\rho^n}{1-\rho} \|x_1 - x_0\|, \qquad n = 0, 1, 2, \ldots. \tag{11}$$

Proof. The iteration process (4) may be written as $x_{n+1} = \Gamma F(x_n)$, $n = 0, 1, 2, \ldots$, where $F(x) = f(0)x - P(x)$. From

$$F'(x) = f(0) - P'(x) = f(0) - f(P(x)) \tag{12}$$

and the Lipschitz continuity of f, it follows that

$$\|F'(x)\| \leqslant \alpha \|P(x)\|, \tag{13}$$

and the theorem follows from (10) and the contraction mapping principle [3].

If P' is Lipschitz continuous in a neighborhood of x^*, then the convergence of the sequence $\{x_n\}$ will be quadratic within this neighborhood as soon as inequality (7) holds with x_0 replaced by an iterate x_n sufficiently close to x^*.

3. A Semilocal Existence Theorem. If f and P are Lipschitz continuous with constants α and γ, respectively, then it follows from (6) that P' is Lipschitz continuous with constant $K = \alpha\gamma$. Furthermore,

$$\|P(x)\| \leqslant \|P(x_0)\| + \gamma\|x - x_0\|. \tag{14}$$

For $r = \|x - x_0\|$, define

$$\rho(r) = \alpha B^*\|P(x_0)\| + B^*Kr. \tag{15}$$

If $\rho(0) = \alpha B^*\|P(x_0)\| < 1$, then inequality (10) and the contraction mapping principle [3, pp. 84–85] give the following result.

THEOREM 3. *If*

$$\Delta = (1 - \alpha B^*\|P(x_0)\|)^2 - 4B^*K\|x_1 - x_0\| \geqslant 0, \tag{16}$$

then a solution x^* of the equation $P(x) = 0$ exists in the closed ball

$$V = \left\{x\colon \|x - x_0\| \leqslant \frac{1 - \alpha B^*\|P(x_0)\| - \sqrt{\Delta}}{2B^*K} = r^*\right\}, \tag{17}$$

and is unique in the open ball

$$U = \left\{x\colon \|x - x_0\| < \frac{1 - \alpha B^*\|P(x_0)\|}{B^*K}\right\}. \tag{18}$$

By itself, the contraction mapping principle only guarantees that

$$\|x_n - x^*\| \leqslant (\rho^*)^n r^*, \qquad n = 0, 1, 2, \ldots, \tag{19}$$

where

$$\rho^* = \rho(r^*) = \tfrac{1}{2}(1 + \alpha B^*\|P(x_0)\| - \sqrt{\Delta}). \tag{20}$$

By Theorem 1, however, the convergence of the sequence $\{x_n\}$ to x^* will be quadratic for $n = N, N + 1, \ldots$, where N is the smallest nonnegative integer satisfying the inequality

$$\theta = \tfrac{1}{2}KB^*(\rho^*)^N r^* < 1. \tag{21}$$

Mathematics Research Center
University of Wisconsin
Madison, Wisconsin 53706

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