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Properties of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Functions

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Abstract. Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions and tables for the first fifteen leading terms are given. Relations of these coefficients with the randomization distributions are shown.

Little is known about the Taylor series expansion coefficients of the Jacobian elliptic functions $\operatorname{sn}(u, k)$, $\operatorname{cn}(u, k)$ and $\operatorname{dn}(u, k)$. No recurrence formula exists for these coefficients. Only four to five leading terms of the series are given in literature ([1], [2]).

We present in this paper properties of these coefficients, show relations between them and randomization distributions [3, p. 51], and give tables for the first fifteen leading terms.

We consider the differential equations

$$\tag{1} \begin{aligned} &\frac{d}{du} y_1(u) - C_1 y_2(u) y_3(u) = 0, \\ &\frac{d}{du} y_2(u) - C_2 y_1(u) y_3(u) = 0, \\ &\frac{d}{du} y_3(u) - C_3 y_1(u) y_2(u) = 0. \end{aligned}$$

Solution functions of (1) for $C_1 = 1$, $C_2 = -1$, $C_3 = -k^2$ are the Jacobian elliptic functions $y_1 = \operatorname{sn}(u, k)$, $y_2 = \operatorname{cn}(u, k)$, $y_3 = \operatorname{dn}(u, k)$ ([1], [2]).

The formal Taylor series of the functions y_1, y_2, y_3 read

$$\tag{2} \begin{aligned} y_1(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum a_{j_1 j_2 j_3} C_1^{j_1} C_2^{j_2} C_3^{j_3} y_{10}^{i_1} y_{20}^{i_2} y_{30}^{i_3} \right], \\ y_2(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum b_{h_1 h_2 h_3} C_1^{h_1} C_2^{h_2} C_3^{h_3} y_{10}^{s_1} y_{20}^{s_2} y_{30}^{s_3} \right], \\ y_3(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum c_{r_1 r_2 r_3} C_1^{r_1} C_2^{r_2} C_3^{r_3} y_{10}^{q_1} y_{20}^{q_2} y_{30}^{q_3} \right]. \end{aligned}$$

The summation over the indices $j_1, j_2, j_3; h_1, h_2, h_3; r_1, r_2, r_3$ and their relation to the exponents of y_{10}, y_{20}, y_{30} are specified in Theorem I.

$$y_{m0} = y_m(u_0) \qquad (m = 1, 2, 3).$$

For $u_0 = 0$, $y_{10} = 0$, $y_{20} = y_{30} = 1$, this series is convergent in the region $|u| < K'$ [2], where

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$$K' = K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - k'^2 \sin^2\theta)}}, \qquad k' = \sqrt{1 - k^2}.$$

For the explicit series of (2), the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$, $c_{r_1 r_2 r_3}$ have to be determined and summation over the indices has to be specified.

THEOREM I. $a_{j_1 j_2 j_3} \neq 0$ *only for*

$$j_1 = 1, 2, \ldots, J_1; \qquad J_1 = \begin{cases} n/2 & \text{for } n \text{ even}, \\ (n+1)/2 & \text{for } n \text{ odd}, \end{cases}$$

$$\begin{matrix} j_2 = 0, 1, \ldots, J_2; \\ j_3 = 0, 1, \ldots, J_3; \end{matrix} \qquad J_2, J_3 = \begin{cases} n/2 & \text{for } n \text{ even}, \\ (n-1)/2 & \text{for } n \text{ odd}, \end{cases}$$

and j_1, j_2, j_3 *satisfying the relation* $j_1 + j_2 + j_3 = n$. i_1, i_2, i_3 *is obtained from the relations* $i_1 = n + 1 - 2j_1$, $i_2 = n - 2j_2$, $i_3 = n - 2j_3$. $b_{h_1 h_2 h_3} \neq 0$ *only for* $h_1 = j_3$, $h_2 = j_1$, $h_3 = j_2$.

The exponents s_1, s_2, s_3 are related to h_1, h_2, h_3 as follows: $s_1 = n - 2h_1$, $s_2 = n + 1 - 2h_2$, $s_3 = n - 2h_3$. $c_{r_1 r_2 r_3} \neq 0$ only for $r_1 = j_2$, $r_2 = j_3$, $r_3 = j_1$. The exponents q_1, q_2, q_3 are defined by the relations $q_1 = n - 2r_1$, $q_2 = n - 2r_2$, $q_3 = n + 1 - 2r_3$.

For $u_0 = 0$, i.e.,

$$y_1(u_0 = 0) = \text{sn}(0, k) = 0, \quad y_2(u_0 = 0) = \text{cn}(0, k) = 1,$$

$$y_3(u_0 = 0) = \text{dn}(0, k) = 1,$$

we obtain

$$a_{j_1 j_2 j_3} \neq 0 \quad \text{only for } j_1 = (n+1)/2; j_2 = 0, 1, \ldots, (n-1)/2; n \text{ odd},$$

$$b_{h_1 h_2 h_3} \neq 0 \quad \text{only for } h_3 = n/2; h_1 = 0, 1, \ldots, n/2 - 1; n \text{ even},$$

$$c_{r_1 r_2 r_3} \neq 0 \quad \text{only for } r_2 = n/2; r_3 = 1, 2, \ldots, n/2; n \text{ even}.$$

In the following table the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$, $c_{r_1 r_2 r_3}$ and the exponents i_1, i_2, i_3; s_1, s_2, s_3; q_1, q_2, q_3 are given as an illustration for $n = 3$ and $n = 4$.

TABLE I

	$a_{j_1 j_2 j_3}$	j_1	j_2	j_3	i_1	i_2	i_3	$b_{h_1 h_2 h_3}$	h_1	h_2	h_3	s_1	s_2	s_3	$c_{r_1 r_2 r_3}$	r_1	r_2	r_3	q_1	q_2	q_3
$n=3$	1	2	1	0	0	1	3	4	1	1	1	1	2	1	4	1	1	1	1	1	2
	1	2	0	1	0	3	1	1	1	2	0	1	0	3	1	1	0	2	1	3	0
	4	1	1	1	2	1	1	1	0	2	1	3	0	1	1	0	1	2	3	1	0
$n=4$	14	2	1	1	1	2	2	4	2	1	1	0	3	2	4	2	1	1	0	2	3
	1	2	2	0	1	0	4	1	2	2	0	0	1	4	1	2	0	2	0	4	1
	4	1	2	1	3	0	2	14	1	2	1	2	1	2	14	1	1	2	2	2	1
	1	2	0	2	1	4	0	4	1	1	2	2	3	0	4	1	2	1	2	0	3
	4	1	1	2	3	2	0	1	0	2	2	4	1	0	1	0	2	2	4	0	1

The following identities and symmetries are valid:

THEOREM II.

$$a_{j_1 j_2 j_3} = b_{j_3 j_1 j_2} = c_{j_2 j_3 j_1},$$
$$a_{j_1 j_2 j_3} = a_{j_1 j_3 j_2};$$

and therefore,

$$b_{j_3 j_1 j_2} = b_{j_2 j_1 j_3},$$
$$c_{j_2 j_3 j_1} = c_{j_3 j_2 j_1}.$$

Theorems III/1, III/2 and III/3 show relations between the randomization distribution and the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$ and $c_{r_1 r_2 r_3}$.

THEOREM III/1. *The sum of all elements $a_{j_1 j_2 j_3}$ for a given n is equal to n!.*

Example: $n = 4$.

$$a_{112} + a_{121} + a_{202} + a_{211} + a_{220} = 4 + 4 + 1 + 14 + 1 = 24 = 4!$$

THEOREM III/2. *For a given $n = 2$, the following relation is valid*:

$$\sum_{j_3=0}^{J_3} a_{j_1 j_2 j_3} \geqslant RU_{j_2}, \qquad j_2 = 0, 1, \ldots, J_2,$$

where RU_{j_2} is the number of permutations of n natural numbers with j_2 runs up. Tables of the numbers RU_{j_2} are given in [3, *p.* 260, *Table* 7.2.2].

Example: $n = 3$.

$$a_{201} = 1 = RU_0, \qquad a_{210} + a_{111} = 1 + 4 = 5 = RU_1.$$

	Permutations	One run up (underlined)	Zero run up (underlined)
	123	$\underline{123}$	123
	132	$\underline{13}2$	132
	231	$\underline{23}1$	231
	213	$2\underline{13}$	213
	312	$3\underline{12}$	312
	321	321	$\underline{321}$
Total	$6 = n!$	$5 = RU_1$	$1 = RU_0$

THEOREM III/3. *For a given $n \geqslant 2$ the following relation is valid*:

$$\sum_{j_2=0}^{J_2} a_{j_1 j_2 j_3} = P_{j_1}, \qquad j_1 = 1, 2, \ldots, J_1,$$

where P_{j_1} is the number of permutations of n natural numbers with $j_1 - 1$ peaks. The *numbers* P_{j_1} *are tabulated in* [3, *p.* 261, *Table* 7.3].

Example: $n = 3$.

$$a_{111} = 4 = P_1, \qquad a_{201} + a_{210} = 1 + 1 = 2 = P_2.$$

	Permutations	One peak (underlined)	Zero peak (underlined)
	123	123	$\underline{123}$
	132	$\underline{132}$	132
	231	$\underline{231}$	231
	213	213	$\underline{213}$
	312	312	$\underline{312}$
	321	321	$\underline{321}$
Total	$6 = n!$	$2 = P_2$	$4 = P_1$

Similar results can be obtained for $b_{h_1h_2h_3}$ and $c_{r_1r_2r_3}$ using Theorem II.

These theorems can be proved by mathematical induction. Theorem III/1 follows from Theorem III/2 or Theorem III/3 since the number of permuations of n natural numbers is equal to $n!$.

Table II, in the microfiche section attached to this issue, lists the elements $a_{j_1j_2j_3}$ for $n = 0, 1, 2, \ldots, 15$.

Putting $u_0 = 0$ the explicit terms of the series for $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ read (Theorem I, Theorem II and Table II):

$$\begin{aligned}
\mathrm{sn}(u, k) = {} & u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - (1 + 135k^2 + 135k^4 + k^6)\frac{u^7}{7!} \\
& + (1 + 1228k^2 + 5478k^4 + 1228k^6 + k^8)\frac{u^9}{9!} \\
& - (1 + 11069k^2 + 165826k^4 + 165826k^6 + 11069k^8 + k^{10})\frac{u^{11}}{11!} \\
& + (1 + 99642k^2 + 4494351k^4 + 13180268k^6 + 4494351k^8 \\
& \qquad + 99642k^{10} + k^{12})\frac{u^{13}}{13!} \\
& - (1 + 896803k^2 + 116294673k^4 + 834687179k^6 + 834687179k^8 \\
& \qquad + 116294673k^{10} + 896803k^{12} + k^{14})\frac{u^{15}}{15!} + \cdots \\
= {} & \sum_{n_0=1;\,(n_0 \text{ odd})}^{\infty} (-1)^{(n_0-1)/2} \left(\sum_{j_2=0}^{(n_0-1)/2} a_{j_1j_2j_3} k^{2j_2} \right) \frac{u^{n_0}}{n_0!}, \quad j_1 = \frac{n_0+1}{2}
\end{aligned}$$

(terms for $n_0 \leqslant 7$ are given in [2]);

$$\operatorname{cn}(u, k) = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!}$$
$$+ (1 + 408k^2 + 912k^4 + 64k^6)\frac{u^8}{8!}$$
$$- (1 + 3688k^2 + 30768k^4 + 15808k^6 + 256k^8)\frac{u^{10}}{10!}$$
$$+ (1 + 33212k^2 + 870640k^4 + 1538560k^6 + 259328k^8 + 1024k^{10})\frac{u^{12}}{12!}$$
$$- (1 + 298932k^2 + 22945056k^4 + 106923008k^6 + 65008896k^8$$
$$+ 4180992k^{10} + 4096k^{12})\frac{u^{14}}{14!} + \cdots$$
$$= 1 + \sum_{n_e=2;\,(n_e \text{ even})}^{\infty} (-1)^{n_e/2}\left(\sum_{h_1=0}^{n_e/2-1} b_{h_1h_2h_3}k^{2h_1}\right)\frac{u^{n_e}}{n_e!}, \qquad h_3 = n_e/2$$

(terms for $n_e \leqslant 8$ are given in [2]);

$$\operatorname{dn}(u, k) = 1 - k^2\frac{u^2}{2!} + (4 + k^2)k^2\frac{u^4}{4!} - (16 + 44k^2 + k^4)k^2\frac{u^6}{6!}$$
$$+ (64 + 912k^2 + 408k^4 + k^6)k^2\frac{u^8}{8!}$$
$$- (256 + 15808k^2 + 30768k^4 + 3688k^6 + k^8)\frac{u^{10}k^2}{10!}$$
$$+ (1024 + 259328k^2 + 1538560k^4 + 870640k^6 + 33212k^8 + k^{10})\frac{u^{12}k^2}{12!}$$
$$- (4096 + 4180992k^2 + 65008896k^4 + 106923008k^6$$
$$+ 22945056k^8 + 298932k^{10} + k^{12})\frac{u^{14}k^2}{14!} + \cdots$$
$$= 1 + \sum_{n_e=2;\,(n_e \text{ even})}^{\infty} (-1)^{n_e/2}\left(\sum_{r_3=1}^{n_e/2} c_{r_1r_2r_3}k^{2(r_3-1)}\right)\frac{u^{n_e}k^2}{n_e!}, \qquad r_2 = n_e/2$$

(terms for $n_e \leqslant 8$ are given in [2]).

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ADDENDUM TO

PROPERTIES OF THE TAYLOR SERIES EXPANSION COEFFICIENTS OF THE JACOBIAN ELLIPTIC FUNCTIONS

(this issue, p. 143)

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TABLE II: The elements $a_{j_1 j_2 j_3}$ for $n = 0, 1, 2, \ldots, 15$ are tabulated. Only the indices j_1, j_2, are explicitly given: $j_3 = n - j_1 - j_2$ (see Theorem 1). RU_{j_2} is the number of permutations of n natural numbers with j_2 runs up and P_{j_1} is the number of permutations of n natural numbers with $j_1 - 1$ peaks. The elements $b_{h_1 h_2 h_3}$ and $c_{r_1 r_2 r_3}$ can be obtained from $a_{j_1 j_2 j_3}$ using Theorem II.

$a_{000}=1; \quad a_{100}=1$

$j_2 \backslash j_3$	0	1	n=2
0		1	1
1	1		1
2			$RU_{i_2} \backslash P_{i_1}$

$j_2 \backslash j_3$	0	1	n=3
0		1	1
1	1	4	5
2	4		$RU_{i_2} \backslash P_{i_1}$

$j_2 \backslash j_3$	0	1	2	n=4
0			1	1
1		14	4	18
2	1	4		5
16	8			$RU_{i_2} \backslash P_{i_1}$

$j_2 \backslash j_3$	0	1	2	n=5
0			1	1
1		14	44	58
2	1	44	16	61
16	88	16		$RU_{i_2} \backslash P_{i_1}$

$j_2 \backslash j_3$	0	1	2	3	n=6
0				1	1
1			135	44	179
2		135	328	16	479
3	1	44	16		61
272	416	32			$RU_{i_2} \backslash P_{i_1}$

$j_2 \backslash j_3$	0	1	2	3	n=7
0				1	1
1			135	408	543
2		135	2064	912	3111
3	1	408	912	64	1385
272	2880	1824	64		$RU_{i_2} \backslash P_{i_1}$

j_2 \ j_3	0	1	2	3	4	n=8
0					1	1
1				1228	408	1636
2			5478	11880	912	18270
3		1228	11880	5856	64	19028
4	1	408	912	64		1385
7936	24576	7680	128			P_{i_1} \ RU_{i_2}

j_2 \ j_3	0	1	2	3	4	n=9
0					1	1
1				1228	3688	4916
2			5478	64920	30768	101166
3		1228	64920	124320	15808	206276
4	1	3688	30768	15808	256	50521
7936	137216	185856	31616	256		P_{i_1} \ RU_{i_2}

j_2 \ j_3	0	1	2	3	4	5	n=10
0						1	1
1					11069	3688	14757
2				165826	343648	30768	540242
3			165826	1146480	621648	15808	1949762
4		11069	343648	621648	96896	256	1073517
5	1	3688	30768	15808	256		50521
353792	1841152	1304832	128512	512			RU_{j_2} \ P_{j_1}

j_2 \ j_3	0	1	2	3	4	5	n=11
0						1	1
1					11069	33212	44281
2				165826	1782800	870640	2819266
3			165826	5429352	9756048	1538560	16889786
4		11069	1782800	9756048	5651456	259328	17460701
5	1	33212	870640	1538560	259328	1024	2702765
353792	9061376	21253376	8728576	518656	1024		RU_{j_2} \ P_{j_1}

j_2 \ j_3	0	1	2	3	4	5	6	n = 12
0							1	1
1						99642	33212	132854
2					4494351	9129868	870640	14494859
3				13180268	78650552	44593984	1538560	37963364
4			4494351	78650552	131469216	26721792	259328	241595239
5		99642	9129868	44593984	26721792	1566208	1024	82112598
6	1	33212	870640	1538560	259328	1024		2702765
22368256	175627264	222398464	56520704	2084864	2048			P_{j_1} \ RU_{j_2}

j_2 \ j_3	0	1	2	3	4	5	6	n = 13
0							1	1
1						99642	298932	398574
2					4494351	46363428	22945056	73802835
3				13180268	350988072	610611072	106923008	1081702420
4			4494351	350988072	1601152704	980493312	65008896	3002137335
5		99642	46363428	610611072	980493312	227870208	4180992	1869618654
6	1	298932	22945056	106923008	65008896	4180992	4096	199360981
22368256	795300864	2868264960	2174832640	357888000	83[illegible]984	4096		P_{j_1} \ RU_{j_2}

j_2 \ j_3	0	1	2	3	4	5	6	7	n = 14
0								1	1
1							896803	298932	1195735
2						116294673	234158760	22945056	373398489
3					834687179	4614655500	2679876768	106923008	8236142455
4				834687179	11318493824	18163000896	4210012160	65008896	34591152955
5			116294673	4614655500	18163000896	11627894784	1050465024	4180992	35576411869
6		896803	234158760	2679876768	4210012160	1050465024	25135104	4096	8200598715
7	1	298932	22945056	106923 .3	65008896	4180992	4096		199360981
1903757312	2016670208	41731645440	20261765120	22309478840	33497088	8192			P_{j_1} \ RU_{j_2}

j_2 \ j_3	0	1	2	3	4	5	6	7	n = 15
0								1	1
1							896803	2690416	3587219
2						116294673	1178416608	586629984	1881341265
3					834687179	20015036304	34199593440	6337665152	61386982075
4				834687179	47310326336	195642911424	123561102848	9860488448	377209516235
5			116294673	20015036304	195642911424	294351274240	81225837312	2536974336	598888328289
6		896803	1178416608	34199593440	123561102848	81225837312	8680206336	67047424	248913100971
7	1	2690416	586629984	6337665152	9860488448	2536974336	67047424	16384	19991512145
1903757312	89702612992	460858269696	559148810240	182172651520	48754155008	134094848	16384		P_{j_1} \ RU_{j_2}

Table 3. (k, ℓ)-methods of highest error order with $\varepsilon > 0$.

k	1										2					
ℓ	1	2		3			4				1	2		3		
ε	1	1	2	1	2	3	1	2	3	4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$
α_0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	$\frac{1}{3}$	$\frac{-7}{23}$	$\frac{-1}{17}$	$\frac{29}{99}$	$\frac{1}{15}$	$\frac{-1}{49}$
α_1	1	1	1	1	1	1	1	1	1	1	$-\frac{4}{3}$	$-\frac{16}{23}$	$-\frac{16}{17}$	$-\frac{128}{99}$	$-\frac{16}{15}$	$-\frac{48}{49}$
α_2											1	1	1	1	1	1
β_{10}	0	$\frac{1}{3}$	0	$\frac{2}{5}$	$\frac{1}{4}$	0	$\frac{3}{7}$	$\frac{1}{3}$	$\frac{1}{5}$	0	0	$\frac{2}{23}$	0	$-\frac{10}{99}$	$\frac{-1}{75}$	0
β_{11}	1	$\frac{2}{3}$	1	$\frac{3}{5}$	$\frac{3}{4}$	1	$\frac{4}{7}$	$\frac{2}{3}$	$\frac{4}{5}$	1	0	$\frac{16}{23}$	$\frac{8}{17}$	$\frac{32}{99}$	$\frac{32}{75}$	$\frac{16}{49}$
β_{12}											$\frac{2}{3}$	$\frac{12}{23}$	$\frac{10}{17}$	$\frac{10}{33}$	$\frac{13}{25}$	$\frac{34}{49}$
β_{20}		0	0	$\frac{1}{20}$	0	0	$\frac{1}{14}$	$\frac{1}{30}$	0	0		0	0	$\frac{-1}{99}$	0	0
β_{21}		$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{-3}{20}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{7}$	$-\frac{1}{5}$	$\frac{-3}{10}$	$-\frac{1}{2}$		0	0	$\frac{16}{99}$	$\frac{8}{75}$	0
β_{22}												$\frac{-2}{23}$	$\frac{-2}{17}$	$\frac{-1}{11}$	$\frac{-8}{75}$	$-\frac{10}{49}$
β_{30}				0	0	0	$\frac{1}{210}$	0	0	0				0	0	0
β_{31}				$\frac{1}{60}$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{2}{105}$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{6}$				0	0	0
β_{32}														$\frac{2}{297}$	$\frac{2}{225}$	$\frac{4}{147}$
β_{40}							0	0	0	0						
β_{41}							$\frac{-1}{840}$	$\frac{-1}{360}$	$\frac{-1}{120}$	$\frac{-1}{24}$						
β_{42}																

Table 3 (continuation)

k	2				3						
l	4				1	2		3			4
ε	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$
α_0	$-\frac{103}{359}$	$\frac{-19}{275}$	$\frac{11}{501}$	$\frac{1}{351}$	$\frac{-2}{11}$	$-\frac{16}{97}$	$\frac{2}{187}$	$\frac{-688}{4333}$	$\frac{29}{2887}$	$\frac{1}{1214}$	$\frac{-8464}{54391}$
α_1	$-\frac{256}{359}$	$-\frac{256}{275}$	$-\frac{512}{501}$	$-\frac{352}{351}$	$\frac{9}{11}$	$-\frac{81}{97}$	$\frac{-27}{187}$	$\frac{8019}{4333}$	$\frac{729}{2887}$	$\frac{81}{607}$	$-\frac{150903}{54391}$
α_2	1	1	1	1	$-\frac{18}{11}$	0	$-\frac{162}{187}$	$-\frac{11664}{4333}$	$-\frac{3645}{2887}$	$-\frac{1377}{1214}$	$\frac{104976}{54391}$
α_3					1	1	1	1	1	1	1
β_{10}	$\frac{38}{359}$	$\frac{26}{1375}$	$\frac{-2}{501}$	0	0	$\frac{4}{97}$	0	$\frac{30}{619}$	$\frac{-6}{2887}$	0	$\frac{13896}{271955}$
β_{11}	$\frac{256}{359}$	$\frac{768}{1375}$	$\frac{64}{167}$	$\frac{128}{351}$	0	$\frac{54}{97}$	0	$-\frac{2916}{4333}$	$\frac{-243}{2887}$	$\frac{-81}{2428}$	$\frac{406782}{271955}$
β_{12}	$\frac{168}{359}$	$\frac{676}{1375}$	$\frac{100}{167}$	$\frac{74}{117}$	0	$\frac{108}{97}$	$\frac{108}{187}$	$-\frac{1458}{4333}$	$\frac{972}{2887}$	$\frac{243}{607}$	$\frac{577368}{271955}$
β_{13}					$\frac{6}{11}$	$\frac{44}{97}$	$\frac{6}{11}$	$\frac{1854}{4333}$	$\frac{1377}{2887}$	$\frac{1209}{2428}$	$\frac{113064}{271955}$
β_{20}	$\frac{26}{1795}$	$\frac{2}{1375}$	0	0		0	0	$\frac{18}{4333}$	0	0	$\frac{1656}{271955}$
β_{21}	$\frac{128}{1795}$	$\frac{128}{1375}$	$\frac{32}{501}$	$\frac{16}{351}$		0	0	$\frac{-729}{4333}$	0	0	$\frac{13122}{54391}$
β_{22}	$\frac{-164}{1795}$	$\frac{-28}{275}$	$\frac{-26}{167}$	$\frac{-62}{351}$		0	0	$\frac{1458}{4333}$	$\frac{486}{2887}$	$\frac{81}{607}$	$\frac{-52488}{271955}$
β_{23}						$\frac{-6}{97}$	$\frac{-18}{187}$	$\frac{-297}{4333}$	$\frac{-252}{2887}$	$\frac{-117}{1214}$	$\frac{-19188}{271955}$
β_{30}	$\frac{4}{5385}$	0	0	0				0	0	0	$\frac{72}{271955}$
β_{31}	$\frac{128}{5385}$	$\frac{64}{4125}$	0	0				0	0	0	$\frac{8748}{271955}$
β_{32}	$\frac{16}{1795}$	$\frac{4}{375}$	$\frac{32}{1503}$	$\frac{28}{1053}$				0	0	0	$\frac{17496}{271955}$
β_{33}								$\frac{18}{4333}$	$\frac{18}{2887}$	$\frac{9}{1214}$	$\frac{1584}{271955}$
β_{40}	0	0	0	0							0
β_{41}	0	0	0	0							0
β_{42}	$\frac{-2}{5385}$	$\frac{-2}{4125}$	$\frac{-2}{1503}$	$\frac{-2}{1053}$							0
β_{43}											$\frac{-54}{271955}$

Table 3 (continuation)

k	3			4					
l	4			1	2		3		
ε	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
α_0	$\frac{196}{20459}$	$\frac{32}{31315}$	$\frac{8}{50293}$	$\frac{3}{25}$	$\frac{-53}{485}$	$\frac{3}{1117}$	$\frac{5031}{47705}$	$\frac{-9}{3385}$	$\frac{-3}{44179}$
α_1	$-\frac{243}{499}$	$\frac{-8019}{31315}$	$\frac{2187}{50293}$	$-\frac{16}{25}$	$-\frac{512}{485}$	$\frac{-64}{1117}$	$-\frac{140288}{47705}$	$\frac{-5888}{37235}$	$\frac{128}{44179}$
α_2	$-\frac{10692}{20459}$	$-\frac{23328}{31315}$	$-\frac{52488}{50293}$	$\frac{36}{25}$	$-\frac{432}{485}$	$\frac{-864}{1117}$	$\frac{373248}{47705}$	$\frac{81648}{37235}$	$\frac{9072}{44179}$
α_3	1	1	1	$-\frac{48}{25}$	$\frac{512}{485}$	$\frac{-192}{1117}$	$-\frac{285696}{47705}$	$-\frac{112896}{37235}$	$-\frac{53376}{44179}$
α_4				1	1	1	1	1	1
β_{10}	$\frac{-160}{61377}$	$\frac{-24}{156575}$	0	0	$\frac{12}{485}$	0	$\frac{-1404}{47705}$	$\frac{18}{37235}$	0
β_{11}	$\frac{3402}{20459}$	$\frac{13122}{156575}$	$\frac{-486}{50293}$	0	$\frac{256}{485}$	0	$\frac{52224}{47705}$	$\frac{1536}{37235}$	0
β_{12}	$\frac{17496}{20459}$	$\frac{110808}{156575}$	$\frac{19440}{50293}$	0	$\frac{864}{485}$	$\frac{432}{1117}$	$-\frac{46656}{47705}$	$-\frac{31104}{37235}$	$\frac{-2592}{44179}$
β_{13}	$\frac{27556}{61377}$	$\frac{72444}{156575}$	$\frac{29136}{50293}$	0	$\frac{768}{485}$	$\frac{1152}{1117}$	$-\frac{82944}{47705}$	$-\frac{18432}{37235}$	$\frac{16128}{44179}$
β_{14}				$\frac{12}{25}$	$\frac{40}{97}$	$\frac{516}{1117}$	$\frac{3744}{9541}$	$\frac{1422}{3385}$	$\frac{21324}{44179}$
β_{20}	$\frac{-4}{20459}$	0	0		0	0	$\frac{-108}{47705}$	0	0
β_{21}	$\frac{486}{20459}$	$\frac{1458}{156575}$	0		0	0	$\frac{9216}{47705}$	0	0
β_{22}	$\frac{972}{20459}$	$\frac{11664}{156575}$	$\frac{3888}{50293}$		0	0	$-\frac{46656}{47705}$	$\frac{-7776}{37235}$	0
β_{23}	$\frac{-1704}{20459}$	$\frac{-13932}{156575}$	$\frac{-7254}{50293}$		0	0	$\frac{27648}{47705}$	$\frac{13824}{37235}$	$\frac{6912}{44179}$
β_{24}					$\frac{-24}{485}$	$\frac{-72}{1117}$	$\frac{-540}{9541}$	$\frac{-2448}{37235}$	$\frac{-3960}{44179}$
β_{30}	0	0	0				0	0	0
β_{31}	0	0	0				0	0	0
β_{32}	$\frac{648}{20459}$	$\frac{3888}{156575}$	0				0	0	0
β_{33}	$\frac{156}{20459}$	$\frac{1332}{156575}$	$\frac{936}{50293}$				0	0	0
β_{34}							$\frac{144}{47705}$	$\frac{144}{37235}$	$\frac{288}{44179}$
β_{40}	0	0	0						
β_{41}	0	0	0						
β_{42}	0	0	0						
β_{43}	$\frac{-6}{20459}$	$\frac{-54}{156575}$	$\frac{-54}{50293}$						
β_{44}									

Table 4. Truncation errors and stability of (k, ℓ)-methods of highest order for a given ε > 0.

k	ℓ	ε	order p	C_{p+1}	Stability
1	1	1	1	-1/2	stable
	2	1	3	1/72	stable
		2	2	1/6	stable
	3	1	5	-1/7200	stable
		2	4	-1/480	stable
		3	3	-1/24	stable
	4	1	7	1/1411200	stable
		2	6	1/75600	stable
		3	5	1/3600	stable
		4	4	1/120	stable
2	1	1/2	2	-2/9	stable
	2	1/2	5	1/1035	stable
		1	4	1/255	stable
	3	1/2	8	-1/1122660	stable
		1	7	-1/189000	stable
		3/2	5	-1/2205	stable
	4	1/2	11	1/3358624500	stable
		1	10	1/428793750	stable
		3/2	8	1/2840670	stable
		2	7	1/442260	stable
3	1	1/3	3	-3/22	stable
	2	1/3	7	3/27160	stable
		2/3	5	3/1870	stable
	3	1/3	11	-3/266912800	unstable
		2/3	9	-3/8083600	stable
		1	8	-1/679840	stable
	4	1/3	15	9/30489418960000	unstable
		2/3	13	1/57342485200	stable
		1	12	3/31346315000	stable
		4/3	9	3/35205100	stable
4	1	1/4	4	-12/125	stable
	2	1/4	9	4/254625	stable(*)
		1/2	7	6/39095	stable
	3	1/4	14	-8/41783616875	unstable
		1/2	12	-4/931805875	unstable
		3/4	9	-4/7731325	stable

(*) This method has the essential root $\zeta_2 = -1$ with the growth parameter $= 1$.