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Decision Problems for *HNN* Groups and Vector Addition Systems

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Abstract. Our purpose is to show the equivalence of the conjugacy problem for certain *HNN* extensions of the infinite cyclic groups and the reachability problem for the class of self-dual vector addition systems. In addition, we extend an endomorphism theorem of the author's to a homomorphism theorem and indicate a problem related to the isomorphism problem for a class of *HNN* groups.

In recent years *HNN* groups and their decision problems have come under extensive investigation [5], [9], [10], [17], [20], [22]. Concurrently, vector addition systems and their decision problems have arisen in several areas of the Computer Sciences [6], [12]–[15], [18], [21]. Here we show the equivalence of the conjugacy problem for certain *HNN* (or strong Britton) extensions of the infinite cyclic group and the reachability problem for the class of self-dual vector addition systems. Generalizations of these results for *HNN* groups with base group as in [5] appear in [4]. In addition, we extend the endomorphism theorem discussed in [1]–[3] to a homomorphism theorem and indicate a problem related to the isomorphism problem for a class of groups arising out of a correspondence between Graham Higman and the author. For concepts and terminology the reader should consult [19]–[21].

Let $HNN(Z)$ denote the groups $G(p_1, q_1, \ldots, p_k, q_k)$ given by

$$\langle a_1, \ldots, a_k, b; a_1^{-1}b^{p_1}a_1 = b^{q_1}, \ldots, a_k^{-1}b^{p_k}a_k = b^{q_k}\rangle \tag{I}$$

where $p_iq_i \neq 0$ for $i = 1, \ldots, k$. We call the integers appearing in (I) the *exponents* of the group. Let l and m be nonzero integers and call m reachable from l with respect to the exponents if there is a sequence of integers beginning with l and ending with m, such that successive terms l_i and l_{i+1} satisfy one of the following conditions:

$$l_{i+1} = l_i(q_j/p_j) \quad \text{where } l_i/p_j \text{ is integral.} \tag{1}$$

$$l_{i+1} = l_i(p_j/q_j) \quad \text{where } l_i/q_j \text{ is integral.} \tag{2}$$

The *reachability problem* for the exponents is to decide for arbitrary such l and m whether m is reachable from l.

Using Collin's Lemma [20, p. 21] and the methods of [5], we may prove:

LEMMA 1. *$G \in HNN(Z)$ has a solvable conjugacy problem if and only if the reachability problem for its exponents is solvable.*

Let (d, W) be an r-dimensional vector addition system and let $R(d, W)$ denote the reachability set of (d, W) [21, p. 292]. The *reachability problem for the set* (*of integral vectors*) W is to decide for arbitrary r-dimensional vectors of nonnegative integers

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d, d' whether $d' \in R(d, W)$. We set $\dim(W) = r$. The dual of W is $-W$ where $w \in W$ if and only if $-w \in -W$ [18, p. 304]. If W coincides with its dual, we call it *self-dual.*

Let VA consist of those groups of $HNN(Z)$ where $p_i, q_i > 0$ and $(p_i, q_i) = 1$ for $i = 1, \ldots, k$. By coding the rational numbers p_i/q_i, q_i/p_i as indicated by [21, p. 294], we can associate an r-dimensional set of intergral vectors $W(p_1, q_1, \ldots, p_k, q_k)$ with the group such that the reachability problem for the set of vectors and the group are equivalent. Moreover, the set is self-dual.

Using concepts known very early (cf. G. Frobenius [11]), J. van Leeuwen reports that, for $\dim(W) \leq 3$, we may decide the reachability problem for any (d, W) [18, Theorem 6.3] and, hence, for W. If the exponents of the group are positive and relatively prime in pairs, then the associated set of vectors is easily seen to have solvable reachability problem (these exponents are called 'unmeshed' in [4]).

From Lemma 1 and the remarks above we conclude:

THEOREM 1. *The conjugacy problem for* $G(p_1, q_1, \ldots, p_k, q_k) \in VA$ *is solvable if and only if the reachability problem is solvable for* $W(p_1, q_1, \ldots, p_k, q_k)$.

Problem 1. Suppose W is self-dual. Need W have a solvable reachability problem?

We extend Theorem 1 of [3] by applying Collin's Lemma.

LEMMA 2. *If* $x \in G(p_1, q_1, \ldots, p_k, q_k)$ *and* x^l *is conjugate to* x^m *for* $|l| \neq |m|$, *then* x *is conjugate to a power of* b.

Let $\hat{G}(p_1, q_1, \ldots, p_k, q_k)$ denote the normal closure of b in $G(p_1, q_1, \ldots, p_k, q_k)$ and let $C(t)$ denote those elements of $\hat{G}(p_1, q_1, \ldots, p_k, q_k)$ which commute with b^t. These groups are tree products of infinite cyclic groups [16] and, in the case of $G(l, m)$ of [1]–[3], [7], [8], turn out to be 'stem' products (which also arise in [22]).

By the *Higman groups* of $HNN(Z)$ we mean those groups such that, for some i, $|p_i| \neq |q_i|$. Let

$$G_1 = G(r_1, s_1, \ldots, r_j, s_j) \quad \text{and} \quad G_2 = G(p_1, q_1, \ldots, p_k, q_k).$$

From Lemma 2 above and Britton's Lemma [18, p. 14] we may extend Theorem 4 of [3].

THEOREM 2. *If* G_1 *and* G_2 *are Higman groups and* $h\colon G_1 \longrightarrow G_2$, $a_i \longrightarrow A_i$, $b \longrightarrow B \neq 1$ *in* G_2 *for* $i = 1, \ldots, j$ *defines a homomorphism, then*

(i) $B = D^{-1}b^tD$, $t \neq 0$,

(ii) $DA_iD^{-1} = c_ix_i$ *where* $c_i \in C(r_it)$ *of* G_2 *and* x_i *is an element of* $\langle a_1, \ldots, a_k\rangle$ *such that* $b^{s_it} = x_i^{-1}b^{r_it}x_i$, $i = 1, \ldots, j$.

Moreover, if h *is an isomorphism, then* $j = k$ *and* $x_1, \ldots, x_k$ *freely generate* $\langle a_1, \ldots, a_k\rangle$.

Let us call t the *parameter* of the homomorphism above. From Theorems 1 and 2 above we conclude:

COROLLARY. *If* G_1 *and* G_2 *are Higman groups,* $G_2 \in VA$ *and* $W(p_1, q_1, \ldots, p_k, q_k)$ *has a solvable reachability problem, then it is decidable whether there exists a homomorphism* $h\colon G_1 \longrightarrow G_2$ *with a given parameter* t.

Problem 2. Let G_1, G_2 be Higman groups. Can one decide if there is a homomorphism $h\colon G_1 \longrightarrow G_2$ with nonzero parameter t?

In light of the recent undecidability results of M. O. Rabin reported in [6] and

M. Hack [13], Problem 2 may prove to be difficult.

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